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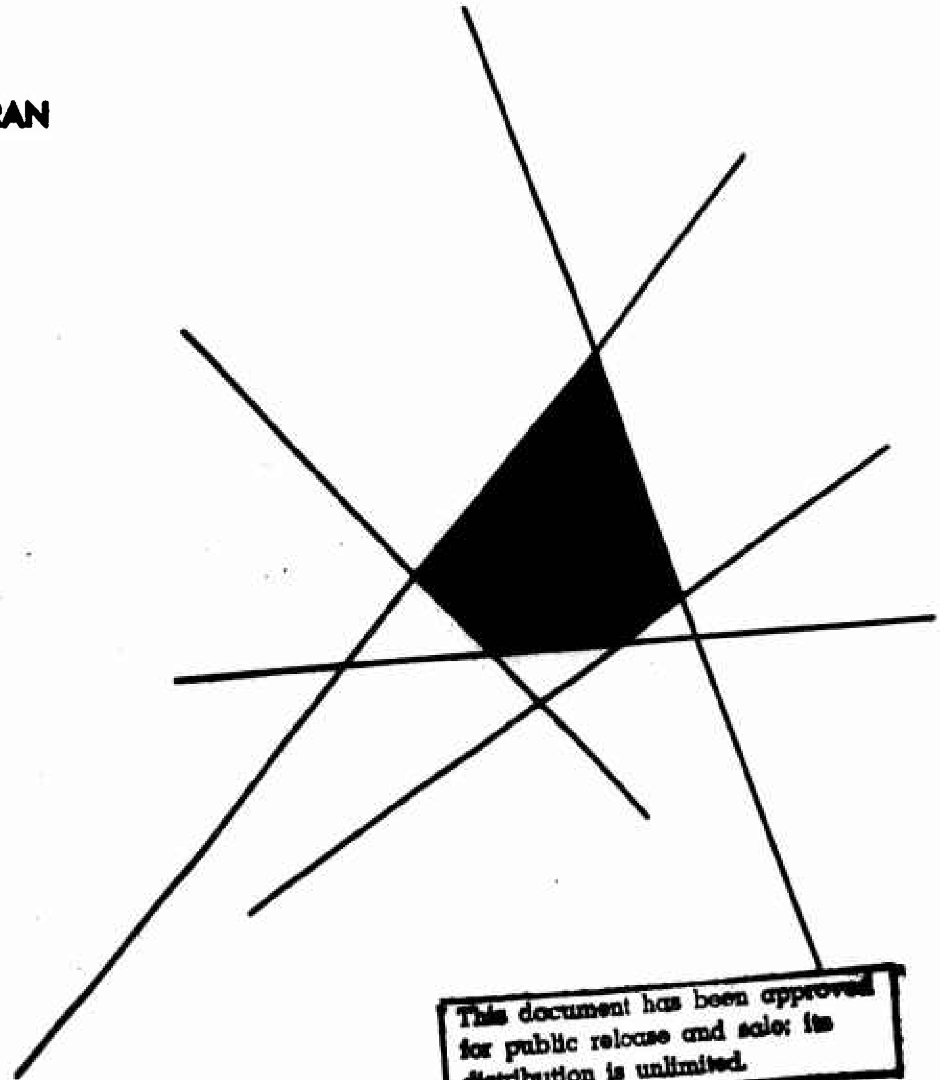
AN INVENTORY MODEL WITH CONTAGIOUS DEMAND DISTRIBUTION

by

ARUNACHALAM RAVINDRAN



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ABSTRACT

In the past, contagious distributions have been successfully applied in Bacteriology, Entomology and Accident Statistics. This thesis applies the notion of contagious distributions in the inventory control of new products and seasonal or style goods, which have an underlying "true contagion" for their demands, viz, the influence of past demands on future occurrence of demands.

A contagious distribution is derived by assuming a modified Poisson process where the demand rate at any instant of time depends on the past demands prior to that instant. A discussion on estimation of the various parameters of the contagious distribution is also included. Using this contagious distribution, a multi-period inventory model is discussed for new product lines with a "fixed periodic review policy." An optimal $s - S$ order policy is derived as a function of the initial stock level and the review period. Seasonal or style goods are treated as single-period inventory problems with contagious demands. An algorithm is developed to compute the optimal order policy and the optimal length of the period.

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INTRODUCTION

In literature, successful applications of contagious distributions are found in Bacteriology, Entomology and Accident Statistics. In this thesis, an attempt has been made to apply the notion of contagious distributions in the inventory control of consumer products which have an underlying "true contagion" for their demands, particularly when the product is new or is a seasonal or style good. "True contagion" means the probability of a favorable event depends on the occurrence of previous favorable events. Thus, unlike the classical Poisson demand, the contagious demand in two nonoverlapping time intervals are dependent as occurrence of a demand influences further occurrence of future demands.

Chapter 1 discusses a "contagious" demand model for an inventory system. Starting with literature review on contagious distributions, the various applications of our contagious demand model are discussed. It is to be noted that the contagious behavior is only transient and once the product stabilizes in the market, the contagion effect vanishes. By assuming a modified Poisson process where the demand rate at any instant of time depends on the number of past demands, the contagious demand distribution is derived. Since the life of new product lines are longer, compared to seasonal goods like clothings, both commodities are treated separately. Seasonal goods are considered as single period inventory problems as the period is small while new product lines are considered as multi-period inventory problems composed of a number of single periods. The contagious demand distributions for succeeding periods for new product lines are also derived.

Chapter 2 discusses the case of new product lines exclusively. Here, a fixed periodic review policy is followed after fixing the review periods institutionally and solving the multi-period problem by successively solving a single period problem, knowing the review period. An $s - S$ policy is derived as an optimal order policy in this case. An algorithm to compute the optimal order level for a given review period is also developed. The optimal ordering policy as a function of the length of the review period is also plotted.

Chapter 3 exclusively discusses the seasonal or style goods which are considered as single period inventory problems. Besides a fixed period-length policy, an optimal period-length policy is also discussed. For the latter case, an algorithm to compute the optimal review period is developed after computing the optimal order level for a given review period. To the author's knowledge, this is the first time an optimization over review period is successfully carried out for an inventory policy under stochastic demand.

Chapter 4 discusses two methods of statistical estimation of the parameters of the contagious demand distribution. Here it is also shown that the maximum likelihood estimator is asymptotically unbiassed and efficient.

Appendices on solution of ordinary differential equations, properties of Beta and Gamma functions which appear in our analysis are also included as reference.

CHAPTER 1

A "CONTAGIOUS" DEMAND MODEL

1.1 Literature Review on Contagious Distributions

In a recently published volume [10], of considerable interest, covering the Proceedings of the International Symposium on Classical and Contagious Discrete Distributions, we observe the successful applications of the notion of contagious distributions, particularly to biological populations, accident statistics, contagious diseases, and psychological data. Actually, the interest on contagious distributions dates back to 1920 when Greenwood and Yule [6] developed a very general scheme for contagious events where the occurrence of each event increases (or decreases) the probability of further events. But due to the very generality of their model, their formulas become too complex for practical applications.

J. Neyman [9] developed and applied successfully three types of contagious distributions in Entomology for distribution of larvae in experimental plots and bacteriology. In a follow up paper by W. Feller [5], it was pointed out that there are two kinds of contagion as "true contagion" and "apparent contagion" and Neyman's [9] contagious distributions are of the latter type.

Apparent contagion is the result of inhomogeneity arising from distributions on the parameters involved in a population. Thus, the compound Poisson distribution is an example of apparent contagion as developed by Greenwood and Yule [6] and applied successfully in accident proneness.

In the case of true contagion, the probability of a "favorable" event depends on the occurrence of previous favorable events. Ironically enough, assuming true contagion, Eggenberger and Polya [3] arrived at the same distribution as obtained by Greenwood and Yule [6]. The Greenwood-Yule-Polya-Eggenberger distribution, which is a negative binomial distribution, has found many applications in contagious diseases, sickness and accident statistics. Gurland [7] discusses a survey of the applications of the negative binomial and other contagious distributions with special reference to some medical data.

1.2 Occurrence of Contagion-Demand in Practice

So far in the literature on contagious distributions, no attempt has been made to apply the notion of contagious distributions in the inventory control of consumer products which have an underlying true contagion for their demands, particularly when the product is new or is a "style" good (which changes its style periodically). The classical demand distribution used in inventory control, like the Poisson distribution, will not reflect the true behavior of the contagious demand. It is well known that the simple Poisson distribution describes mutually independent events; in other words, with a Poisson distribution the number of events in two nonoverlapping time intervals are uncorrelated and the occurrence of an event has no influence on the probability of occurrence of further events.

If we study the demand for the new products and "style" goods, we will note that besides the constant demand for the products (which is mainly due to advertisement), a contagious demand also occurs due to the customers who have used the product and recommended it to their friends or to other sources of consumer awareness. So until the new product is stabilized in the market, there is a "contagious" demand during the initial or "transient" stage, besides the constant demand. Unlike Poisson demand, the contagious demand in two nonoverlapping time intervals are dependent as occurrence of a demand influences further occurrence of future demands. Given that a demand has occurred for a new product from a particular area, we expect to find more demands to come from the same area due to the influence of the past occurred demand. In other words, our demand model has the underlying assumption that past demands have an influence on the occurrence of future demands.

1.3 Applications of Contagion-Demand

A number of examples of products can be thought of, which will follow a contagious law for their demand. For example, acquiring a Princess telephone in your home will tempt your neighbor to do the same. Other examples may be, demands for new cereals, new freeze-dried coffee, new books, cars (style goods), Christmas trees and practically all consumer products in daily use. The notion of contagion can even be extended to nonconsumer products which exhibit a contagion pattern like research reports. In other words, a research organization can do a better control of their inventory of reports assuming a contagious demand distribution for their reports. Also, the sequence of published research papers in a particular subject, will tend to follow a contagious law. This should be so as readers after seeing a research paper tend to work more in the same area which results in more papers. This may be useful to a librarian or publisher of a technical journal. Another classical application is the efficient usage of hospital beds for patients with contagious disease.

1.4 A Contagious Demand Model

The demand model assumes that every occurrence of demand produces one unit of demand implying the demand has a discrete probability distribution. It was discussed in Sections (1.2) and (1.3) how the contagious demand arises in practice and its various applications. To incorporate this idea in our demand model, we follow Cox and Miller [1] by assuming a constant demand rate and superimposed on that a unit contagious demand rate to reflect the influence of each past demand. So at any point in time "t," the total contagious demand rate will be equal to the unit contagious demand rate times the past demands before "t." So the rate at which demand will occur at any time "t," will be the sum of constant demand rate and the total contagious demand rate. Thus, we also introduce, explicitly, time as a parameter in our demand model. Though, the final contagious demand distribution, like many other contagious distributions, is negative binomial in some sense, a derivation of the distribution will be carried out in the analysis to follow, to introduce "time" explicitly since "time" never appears as an explicit parameter in classical negative binomial distributions. (The reader is referred to Feller [4] for a discussion on classical negative binomial distribution.)

Some caution must be applied as to the duration and magnitude of contagious demand. One cannot expect the same amount of contagious demand throughout the life of a product. As a matter of fact, the contagious demand will be very high when the product has just been introduced in the market and will follow a decay law such that its effect vanishes after a time the product is stabilized in the market. So the unit contagion rate, which reflects the increase in the constant demand rate, is a decreasing function of time.

1.5 Periodic Review Policy

There are two types of phenomenal situations that arise in practice. One is the consideration of new product lines like new cereals, while the other is the consideration of seasonal or style goods like automobiles and clothes. If we assume that the contagion rate varies at each instant of time, this will complicate our analysis and results considerably. So, a simplifying approximation is made about the contagion rate for the two class of goods which exhibit a true contagion for their demand.

Most of the seasonal or style goods change annually or semi-annually. So the length of time they remain in market is much smaller compared to that of new product lines. Hence, an approximation is made by using an (average) constant contagion rate throughout the season. This may be valid if the season is small relative to the variation of the contagion rate or if the contagion rate is changing very slowly.

In the case of new products which will be in the market for a considerable length of time, the contagion rate is approximated as a step function with breaks at regular intervals of time identified as the length of the review period T. Choice of T is made institutionally by observing how rapidly the contagion rate is changing. The inventory system is reviewed periodically and at the end of every review period, a decision is made about the best stock up level for the succeeding period depending on the present inventory level, i.e., a Periodic Review Policy is employed. The unit contagion rate is not assumed to be the same from one period to another. Generally, the contagion rate will be less in each succeeding period and the successive new value at the end of every review period is estimated by the knowledge of the past realizations of demand. Under this assumption, the unit contagion rate at time t, denoted by $\alpha(t)$, looks as follows under a particular realization.

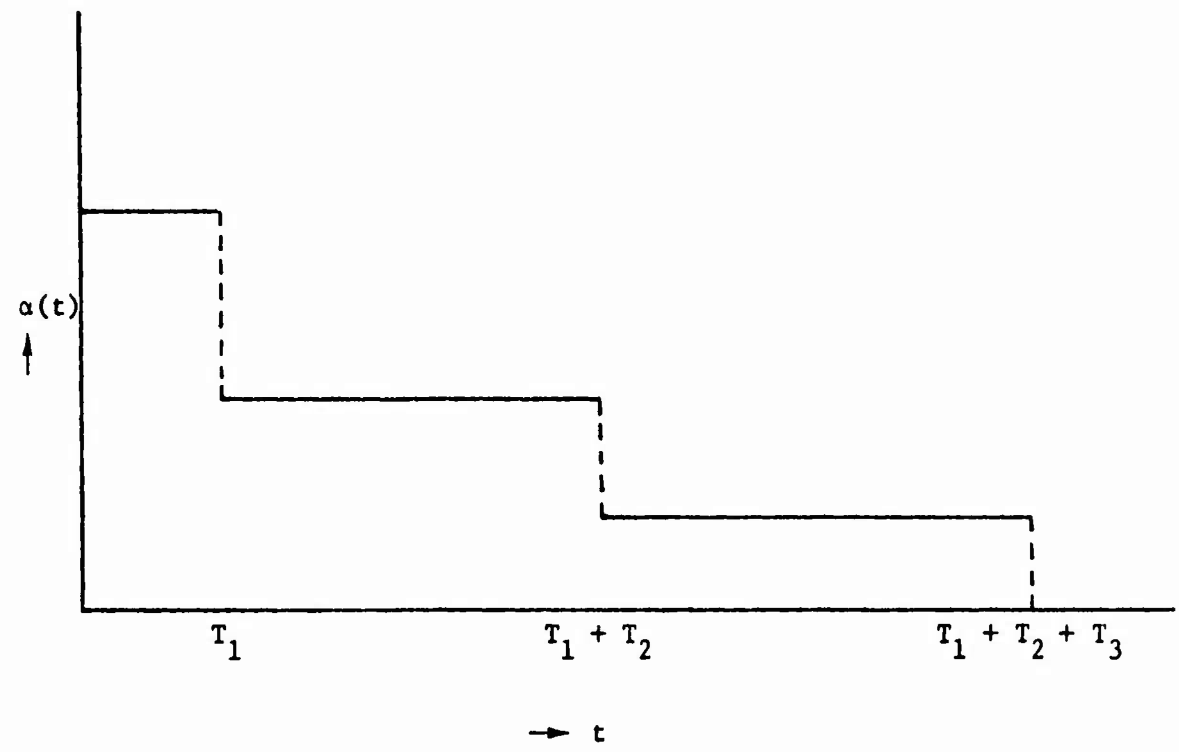

FIGURE 1.1

In the above illustration, the product has three review periods during its transient stage, viz, T_1 , T_2 and T_3 . At the end of time $T_1 + T_2 + T_3$, the product is stabilized in the market and the contagion effect vanishes.

1.6 Assumptions and Notations

Without loss of generality, the beginning of each period is taken as time zero. For the present, we will confine ourselves to seasonal goods or to the first review period of new product lines, where at time zero the demand is zero.[†]

[†]This is not true for succeeding periods of new product lines as the demands in the previous periods will influence the demand distribution in the succeeding periods. Though this does not complicate our analysis too much (luckily!), we postpone its discussion to Section 1.9.

Let $\lambda > 0$ denote the initial constant demand rate component. The unit contagious demand rate is denoted by $\alpha > 0$. In other words, α is the increase in demand rate for each past demand. $N(t)$ denotes the number of demands in an interval $[0,t]$ of length t. The probability of n demands in $[0,t]$ is denoted by $P_n(t)$. Let T denote the length of the review period.

Denoting by h, the length of a very small interval, we make the following assumptions:

Probabilities of positive demand occurrence in the interval $(t,t+h)$ given r previous demands in $(0,t)$ satisfy:

(1.1) $P\{N(t+h) - N(t) = 1/N(t) = r\} = (\lambda + \alpha r)h + o(h)$ where $r = 0,1,2,\ldots$

and $o(h)$ denotes higher order terms in h such that $\lim\limits_{h\to 0} \frac{o(h)}{h} = 0$.

(1.2) $P\{N(t+h) - N(t) \geq 2/N(t) = r\} = o(h)$.

(1.3) $P\{N(t+h) - N(t) = 0/N(t) = r\} = 1 - (\lambda + \alpha r)h + o(h)$.

1.7 The Contagious Probability Distribution for Demand

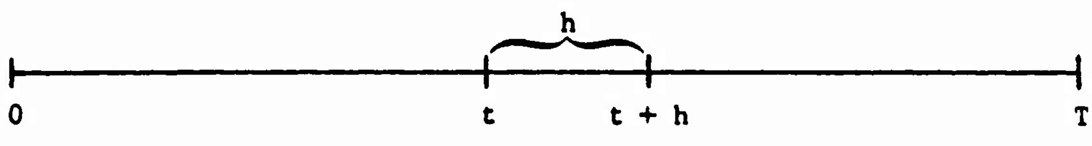

FIGURE 1.2

In the above illustration of the review period $[0,T]$, consider an instant of time $t \in [0,T]$. Assume the interval $[t,t+h]$ of length h satisfies the Properties (1.1), (1.2) and (1.3). Hence, we can write the following expression for the probability $P_n(t+h)$ in terms of probabilities at time t for $n = 0,1,2, \ldots$.

For $n = 0$:

$$P_o(t + h) = P_o(t)[1 - \lambda h + o(h)] .$$

Taking $P_o(t)$ to the left and dividing by h , this reduces to

$$\frac{P_o(t + h) - P_o(t)}{h} = -\lambda P_o(t) + \frac{o(h)}{h} .$$

Taking the limit as h tends to 0 , we get

$$P_o'(t) = -\lambda P_o(t) .$$

Integrating both sides and using the initial condition $P_o(0) = 1$, we get

$$P_o(t) = e^{-\lambda t} .$$

Similarly, for all $n \geq 1$:

$$\begin{aligned} P_n(t + h) = &\text{Prob } [N(t) = n \text{ and no demand in } (t,t + h)] \\ &+ \text{Prob } [N(t) = n - 1 \text{ and one demand in } (t,t + h)] \\ &+ \text{Prob } [N(t) = n - k \text{ and } k(\geq 2) \text{ demands in } (t,t + h)] . \end{aligned}$$

Writing down the corresponding probabilities and using 1.1, 1.2 and 1.3, we get

$$\begin{aligned} P_n(t + h) = &P_n(t)[1 - (\lambda + \alpha n)h] \\ &+ P_{n-1}(t)[\lambda + \alpha(n - 1)]h + o(h) . \end{aligned} \tag{1.4}$$

Taking $P_n(t)$ to the left and dividing by h , (1.4) reduces to

$$\frac{P_n(t+h) - P_n(t)}{h} = -(\lambda + \alpha n)P_n(t) + [\lambda + \alpha(n-1)]P_{n-1}(t) + \frac{o(h)}{h} .$$

Taking the limit of both sides as h tends to 0, we get

$$(1.5) \qquad P_n'(t) = -(\lambda + \alpha n)P_n(t) + [\lambda + \alpha(n-1)]P_{n-1}(t) .$$

Define:

$$(1.6) \qquad \begin{aligned} P_{-1}(t) &\triangleq 0 && \text{for all } t \geq 0 \\ P_o(0) &= 1 \\ P_n(0) &= 0 && \text{for all } n > 0 . \end{aligned}$$

The solution of the above differential Equation (1.5) is

$$(1.7) \qquad \begin{aligned} P_n(t) = e^{-(\lambda+n\alpha)t}[\lambda + \alpha(n-1)] \int e^{(\lambda+n\alpha)t}P_{n-1}(t)dt \\ + ce^{-(\lambda+n\alpha)t} . \end{aligned}$$

The general solution holds for all values of $n = 1,2, \ldots$ and the constant of integration c can be evaluated using the initial conditions given in (1.6). It has already been derived that

$$\begin{aligned} P_o(t) &= \text{Prob. of no demand in } (0,t) \\ &= e^{-\lambda t} . \end{aligned}$$

Solving (1.7) for $n = 1$, we get

$$P_1(t) = \frac{\lambda}{\alpha} e^{-\lambda t} + ce^{-(\lambda+\alpha)t} .$$

As $t \to 0$, $P_1(t) \to 0$ which gives $c = -\lambda/\alpha$. Hence, we get the probability of one demand in $(0,t)$ as

$$P_1(t) = \frac{\lambda}{\alpha} e^{-\lambda t}[1 - e^{-\alpha t}] .$$

Similarly, by substituting $n = 2$ and $n = 3$, and solving for c using (1.6), we get the following probability expressions:

$$P_2(t) = \frac{\lambda(\lambda + \alpha)}{2!\alpha^2} e^{-\lambda t}[1 - e^{-\alpha t}]^2$$

and

$$P_3(t) = \frac{\lambda(\lambda + \alpha)(\lambda + 2\alpha)}{3!\alpha^3} e^{-\lambda t}[1 - e^{-\alpha t}]^3 .$$

Extending the results to n, we get

$$P_n(t) = \frac{\lambda(\lambda + \alpha) \ldots [\lambda + (n - 1)\alpha]}{n!\alpha^n} e^{-\lambda t}[1 - e^{-\alpha t}]^n .$$

Taking the factor α out from each term, we get

$$(1.8) \quad P_n(t) = \frac{1}{n!} \cdot \frac{\lambda}{\alpha} \cdot \left(\frac{\lambda}{\alpha} + 1\right) \cdots \left(\frac{\lambda}{\alpha} + (n - 1)\right) e^{-\lambda t}[1 - e^{-\alpha t}]^n .$$

Define $\frac{\lambda}{\alpha} = \rho$ (a constant). Since λ and α are positive, ρ is also positive. But, ρ need not necessarily be an integer. Using the substitution $\lambda/\alpha = \rho$, we can rewrite (1.8) as

$$P_n(t) = \frac{(\rho + n - 1)(\rho + n - 2) \ldots (\rho + 2)(\rho + 1)\rho}{n!} (e^{-\alpha t})^\rho (1 - e^{-\alpha t})^n .$$

Using the well-kr ı gamma notation, we can write

$$(1.9) \qquad P_n(t) = \frac{\Gamma(\rho + n)}{\Gamma(\rho)\Gamma(n + 1)} (e^{-\alpha t})^\rho (1 - e^{-\alpha t})^n .$$

To simplify the writing, we will henceforth denote $\frac{\Gamma(\rho + n)}{\Gamma(\rho)\Gamma(n + 1)}$ as

$\binom{\rho + n - 1}{n}$. Note:

$$\Gamma(n + 1) = n\Gamma(n) \; ; \; \Gamma(\tfrac{1}{2}) = \sqrt{\pi} \; .$$

Thus, the contagious distribution becomes:

$$(1.10) \qquad P_n(t) = \binom{\rho + n - 1}{n}(e^{-\alpha t})^{\rho}(1 - e^{-\alpha t})^n \qquad \text{for all} \quad n = 0,1,2, \ldots$$

Equation (1.10) can also be interpreted in some sense as a negative binomial distribution by suppressing time. Then $P_n(t)$ will be the probability that exactly n failures precede the ρth success, if ρ is an integer where prob. of success is given by $p = e^{-\alpha t} \geq 0$ and prob. of failure $= q = 1 - e^{-\alpha t} \geq 0$ for all $t \geq 0$, $\alpha > 0$. Hence, $P_n(t)$ can be rewritten as

$$(1.11) \qquad P_n(t) = \binom{\rho + n - 1}{n} p^{\rho} q^n \qquad \text{for all} \quad n = 0,1,2, \ldots$$

Note: The Equations (1.10) and (1.11) hold for $n = 0$ also, as from (1.9),

$$P_o(t) = \frac{\Gamma(\rho + 0)}{\Gamma(\rho)\Gamma(0 + 1)} p^{\rho} = p^{\rho} = e^{-\lambda t} \; .$$

Following Feller [4], we can rewrite the binomial coefficients and using the fact for any $a > 0$

$$\binom{-a}{k} = (-1)^k \binom{a + k - 1}{k} \, ,$$

we get

$$(1.12) \qquad P_n(t) = \binom{-\rho}{n} p^{\rho} (-q)^n$$

where

$$\binom{-\rho}{n} = (-1)^n \frac{\Gamma(\rho + n)}{\Gamma(\rho)\Gamma(n + 1)} .$$

Tables of the negative binomial distribution are available in [14]. Also, Taylor [13] shows an important mathematical equality in his paper that the infinite sum of negative binomial terms can be expressed as a finite sum of (positive) binomial terms. Hence, the latter's table can be used to find the former.

Feller [4] has also shown that Equations (1.11) and (1.12) represent an honest probability distribution. Hence, for a fixed "t," (1.10) also represents an honest probability distribution.

By definition the generating function of $P_n(t)$, denoted by $G_t(Z)$, will be, $G_t(Z) = \sum_{n=0}^{\infty} Z^n P_n(t)$. It can be easily verified that

$$(1.13) \quad G_t(Z) = \left(\frac{p}{1 - qZ}\right)^{\rho} \quad \text{where} \quad p = e^{-\alpha t} , \; q = 1 - e^{-\alpha t} .$$

1.8 Central Moments of the Contagious Distribution

Using the moment generating function, the mean or expectation of demand, for fixed t , will be, by definition

$$m(t) \triangleq \left. \frac{\delta G_t(Z)}{\delta Z} \right|_{Z=1} .$$

Using (1.13),

$$m(t) = \left. \frac{\delta}{\delta Z} \left[\frac{p}{1 - qZ}\right]^{\rho} \right|_{Z=1}$$

$$= \left. \left[\rho\left(\frac{p}{1 - qZ}\right)^{\rho-1} \frac{pq}{(1 - qZ)^2}\right] \right|_{Z=1} .$$

Hence,

$$m(t) = \frac{\rho q}{p} = \frac{\rho(1 - e^{-\alpha t})}{e^{-\alpha t}} \quad (1.14)$$

Similarly, taking the second derivative of (1.13) with respect to Z and taking the limit as $Z \to 1$, we get

$$\sum_{n=0}^{\infty} n(n - 1)P_n(t) = \frac{\rho(\rho + 1)q^2}{p^2} .$$

By definition, the variance is given by

$$V(t) = \sum_{n=0}^{\infty} n^2 P_n(t) - [m(t)]^2 .$$

Hence,

$$V(t) = \frac{\rho q}{p^2} = \frac{\rho(1 - e^{-\alpha t})}{e^{-2\alpha t}} \quad (1.15)$$

1.9 Demand Distribution in Second and Succeeding Periods

As pointed out in Section 1.6 in the case of new product lines, the analysis for first period and succeeding periods differ mainly because of the influence of first period demands on the second and succeeding periods, due to the contagious effect. Luckily enough, the analysis is not complicated too much, and we get the same form of the contagious distribution as derived for the first period, except for a change in parameter depending on the number of demands in the previous periods. The reader will immediately notice that the analysis for the second period will follow identically to third and succeeding periods. The demand distribution for the second period is derived assuming the number of demands in the first period to be N_1 and the new estimated value of the contagious factor be α_2 . Once again the beginning of the review period will be denoted as zero.

Referring to Figure 1.2, probability of a demand in the interval $(t,t+h)$ given r demands in $(0,t)$ will be

$$(1.16) \quad P\{N(t+h) - N(t) = 1/N(t) = r\} = (\lambda + \alpha_2 N_1 + \alpha_2 r)h + o(h) .$$

Since we know α_2 and N_1 , we know the product $\alpha_2 N_1$ which is a "constant" demand rate and can be added the other constant demand rate λ . Let

$$\lambda_2 = \lambda + \alpha_2 N_1 .$$

Hence, (1.16) reduces to

$$(1.17) \quad P\{N(t+h) - N(t) = 1/N(t) = r\} = (\lambda_2 + \alpha_2 r)h + o(h)$$

once again

$$(1.18) \qquad P\{N(t+h) - N(t) \geq 2/N(t) = r\} = o(h)$$

and

$$(1.19) \quad P\{N(t+h) - N(t) = 0/N(t) = r\} = 1 - (\lambda_2 + \alpha_2 r)h + o(h) .$$

Immediately, we notice that the simple substitution $\lambda_2 = \lambda + N_1\alpha_2$, has reduced the Equations (1.17), (1.18) and (1.19) identical to (1.1), (1.2) and (1.3). Hence, we will get the same differential Equation (1.5) except that λ will be replaced by λ_2 , which will lead to identical solution for $P_n(t)$. Denoting $\frac{\lambda_2}{\alpha_2} = \rho_2$, we get

$$(1.20) \qquad P_n^{(2)}(t) = \frac{\Gamma(\rho_2 + n)}{\Gamma(\rho_2)\Gamma(n+1)} \left(e^{-\alpha_2 t}\right)^{\rho_2} \left(1 - e^{-\alpha_2 t}\right)^n \quad \text{for all } n = 0,1,2, \ldots$$

Note that we can extend these results to any succeeding period. For example, let α_N denote the unit contagion rate for the Nth period. Then $\lambda_N = \lambda + \sum_{i=1}^{N-1} N_i\alpha_N$ where N_i is the number of demands in the ith period. We can define ρ_N similarly as $\rho_N = \frac{\lambda_N}{\alpha_N}$, and we will have

$$P_n^{(N)}(t) = \frac{\Gamma(\rho_N + n)}{\Gamma(\rho_N)\Gamma(n + 1)} \left(e^{-\alpha_N t}\right)^{\rho_N}\left(1 - e^{-\alpha_N t}\right)^n \quad \text{for all} \quad n = 0,1,2, \ldots$$

Thus, we notice that the probability distribution for demands in any review period depends on the number of demands that occurred in previous periods. It is interesting to note how the central moments vary. We shall confine ourselves to the second period as the results are identical for succeeding periods.

$$m_2(t) = \text{mean number of demands in 2nd period}$$

$$= \frac{\rho_2\left(1 - e^{-\alpha_2 t}\right)}{e^{-\alpha_2 t}} .$$

Since ρ_2 increases linearly as the number of demands in the first period (N_1) increases, we see that $m_2(t) \uparrow N_1$, and so does, $V_2(t)$, the variance, as

$$V_2(t) = \frac{\rho_2\left(1 - e^{-\alpha_2 t}\right)}{e^{-2\alpha_2 t}} .$$

1.10 Limiting Distribution

Since it is postulated that the contagion rates $\alpha_1, \alpha_2, \ldots$ are strictly decreasing, it will be important to know the limiting form of the contagious distribution as the contagious effect vanishes. It turns out the limiting distribution is Poisson.

Theorem 1.1:

The limiting distribution of $P_n(t)$, given by Equation (1.10), tends to a Poisson law, as $\alpha \to 0$.

Proof:

To prove that Poisson is the limiting form of $P_n(t)$, we follow Feller's [4] approach, using the generating function. From (1.13), the generating function of $P_n(t)$ is given by

$$G_t(Z) = \left(\frac{p}{1 - qZ}\right)^{\rho}$$

where

$$p = e^{-\alpha t}$$

$$p + q = 1$$

$$\rho = \lambda/\alpha .$$

As $\alpha \to 0$; $\rho \to \infty$, $p \to 1$ and $q \to 0$; let $\rho q \to \lambda t$ (fixed). Taking the limit of $G_t(Z)$ as $\alpha \to 0$, we get

$$\lim_{\alpha \to 0} G_t(Z) = \lim_{\rho \to \infty} \left[\frac{1 - \lambda t/\rho}{1 - \lambda Z t/\rho}\right]^{\rho} = \lim_{\rho \to \infty} F(\rho)$$

where

$$F(\rho) = \left[\frac{1 - \lambda t/\rho}{1 - \lambda Z t/\rho}\right]^{\rho} .$$

Taking logarithms of both sides, we get

$$\lim_{\rho\to\infty} \log_e F(\rho) = \lim_{\rho\to\infty} \left[\frac{\log(1 - \lambda t/\rho) - \log(1 - \lambda Z t/\rho)}{1/\rho}\right] .$$

Applying L'Hospital's Rule, we get

$$\lim_{\rho\to\infty} \log F(\rho) = -\lambda t + \lambda Z t .$$

Hence, the limiting value of the generating function as $\alpha \to 0 = e^{-\lambda t(1-Z)}$ which is nothing but the generating function of a Poisson distribution. Hence, as

$$\alpha \to 0 , \; P_n(t) \to \frac{e^{-\lambda t}(\lambda t)^n}{n!} .$$

Note:

By our assumption, in a multi-period model, the contagion rates are strictly decreasing with each period and tends to zero after a finite number of periods.

Hence, ρ_N increases with N and tends to $+\infty$ in a finite period. From the proof of Theorem 1.1, it can be observed that $P_n^{(N)}(t)$ tends to a Poisson distribution after a finite number of periods.

1.11 The Inventory System

In Section 1.5, it was shown that an approximation is made about the contagion rate by following a periodic review policy, i.e., reviews of the inventory system at stated intervals of time and depending on the inventory level realized at the beginning of each interval, the best ordering policy is chosen for the succeeding period. Thus, the periodic review policy leads to a multi-period inventory problem. Because of the computational difficulty of not having an explicit expression for the estimates[†] of the contagion factor, the problem cannot be treated as a multi-period problem but only as successive single period problems. At the beginning of each interval, using the past realization of demand, the new value of the contagion factor is estimated which gives the contagious demand distribution for that interval.

So far, nothing has been assumed about the stated intervals of time. There are number of ways to determine these intervals of time. One way is to fix this review period institutionally either by the knowledge of past experience or arbitrarily. In this case, the review periods may be of increasing length in the order of the period, to facilitate looking at the system more often during the initial periods, which one might normally do since the "learning" about the demand behavior is limited during the initial periods. This type is a fixed periodic review policy. Chapter 2 essentially discusses an optimal order level for a fixed periodic review policy for a single period problem.

Again, it was shown in Section 1.5 that for the case of seasonal or style goods, we only have a single period to consider. To determine this length of the review period T, one may follow the fixed period-length policy by choosing T by past experience or arbitrarily. Another way would be to seek the "best" review period to choose. This leads to an optimization problem with two variables, namely, an order level and a review period. This type is an optimal period-length policy, which can be solved as a "two-stage optimization" problem. The optimization proceeds by first finding the optimal order policy as a function of the length of the review period and then selecting an optimal review period. By considering the optimization in two stages, an optimal solution for a fixed period-length

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[†]Chapter 4 discusses the estimation procedures for the parameters of the contagious distribution.

policy is also given. This is discussed in Chapter 3. This is the first time, to the author's knowledge, an attempt has been made successfully to find an optimal review period for an inventory policy under stochastic demand. For a deterministic demand, we have the classical result in the Wilson-Harris+ lot size formula.

Two situations arise in an inventory system from demands which occur when the inventory is zero:

(i) The demand is backordered and supplied at the start of the next period, i.e., backorders are filled.

(ii) The demand is not filled, i.e., it is lost.

Both the above mentioned situations can arise for seasonal goods as well as new product lines

1.12 Costs Associated with the Inventory System

Three types of costs are considered: (i) procurement costs, (ii) inventory holding cost, and (iii) stock-out costs.

One may observe in inventory literature that the cost function is assumed convex without specifically giving the expression for it. Instead, in this thesis, each of the costs are purposely discussed and an acceptable parametric form is given. In practice, this facilitates computing these costs more easily as one has to only compute the various parameters. Also, this gives an opportunity to make a parametric study of the optimal solution and see how sensitive is the optimal solution with respect to the various cost parameters.

+A detailed discussion of Wilson-Harris lot size formula can be found in Hadley [8].

1.13 Procurement Costs

We shall begin by examining the procurements costs, which can be divided into two parts. First, there is the cost of the goods itself which is paid to source of production. Then there is the costs incurred by the inventory system itself in making a procurement. These may include costs associated with ordering, bookkeeping, transporting, inspection, testing and so on. These themselves fall into two classes, one which depends on the amount ordered and the other independent of the amount ordered. Including the cost of the goods itself in the first classification, we can denote the procurement costs as the sum of two costs; one, which depends on the amount ordered, denoted by c/unit ordered, and, the other, independent of the amount ordered, denoted by k, called the set-up cost. Hence, the total cost of placing an order for Q units will then be $k + cQ$.

1.14 Inventory Holding Costs

The next important cost to consider is the inventory holding cost. Included in these are the real out of pocket costs such as cost of insurance, taxes, breakage and pilferage at storage site, warehouse rental and costs of operating the warehouse. But, the most important cost is not the direct out of pocket cost but an "opportunity cost" which is the cost incurred by having the capital tied up in inventory rather than having it invested elsewhere, and it is equal to the largest rate of return which the system could obtain from alternate investments. So, we shall assume that the instantaneous rate at which inventory carrying charges are incurred are proportional to the investment in inventory at that point in time. Let, I, denote the holding cost in \$/unit time/\$ invested in inventory. According to Hadley [8], the reasonable real world values for I range from something like 0.15 to 0.35.

1.15 Stock-Out Cost

Back-order costs are difficult to measure since they can include such factors as loss of customers' goodwill (i.e., in the future, he may take his business elsewhere). Two types of back order cost for our inventory system are considered:

(i) A back order or shortage cost depending only on the amount backordered. Denote this by p/unit backordered. This may include the cost of notifying the customer, bookkeeping of the amount of back orders and so on.

(ii) A variable cost depending on the length of time for which an order remains unfilled, e.g., a machine shop where lack of parts keeps the machine idle. Denote this by $\hat{p}$/unit short/unit time in back order.

1.16 Selection of an Operating Order Policy

The purpose of this chapter is to find an "optimal" ordering policy with the help of a mathematical model of the inventory system. So far, the term "optimal" has been used loosely. By an "optimal" policy we mean, the ordering policy that maximizes the net profit or minimizes the total costs. In some situations, like the ones we are interested, viz, production of consumer products, one would like to maximize one's net profit. In some other situations, where the profits are always negative (e.g., post office), one may like to minimize the cost. In some cases, both these considerations may arrive at the same ordering policy. One should remember that the net profit or total costs need not be equivalent to a strict accounting profit or cost, since for purposes of computing optimal ordering policy it is only necessary to include those costs which vary with the operating policy. Costs which are independent of the operating doctrine, like the cost of operating the information processing system (which includes the cost of making an actual inventory count, use of computer or the cost of making demand predictions), need not be included. There is another reason why the profit or cost will differ from what would be computed from accounting records. This is because the stock-out costs include components which are not out of pocket costs, like the opportunity costs.

CHAPTER 2

SINGLE PERIOD OPTIMAL ORDER POLICY

2.1 Feed Back Control Policy

It was shown in Chapter 1 that the multi-period inventory problem for the case of new product lines is solved by considering it as successive single period problems. Thus, this chapter will be devoted exclusively for finding an optimal order level for a single period problem under a fixed periodic review policy where the demand for the new products follow a contagious law. Because of the fixed periodic review policy, the review periods are chosen by the knowledge of past experience or arbitrarily.

As discussed earlier, the inventory model calls for new estimated values of the parameters of the demand distribution after every review period. Hence, it is assumed that the values of the constant demand rate (λ) , the unit contagion rate (α) , the number of demands in the previous periods $(\sum N_i)$ and finally, the review period (T) are known. This will completely specify the contagious distribution, given by Equation (1.10).

Since a single period problem is being solved at the end of any review period, the past experience of the demands in the previous periods is used for the succeeding periods. Thus, an efficient feed-back control policy may be determined as information is constantly fed back into the inventory system and this feed back is used in determining the ordering policy successively. There is also a "learning process" associated with the operation of the inventory system.

2.2 Notations

We shall review all our notations and symbols used so far and introduce a few more in this section.

x - initial inventory level before reordering goods.

y - starting inventory level after reordering. (Hence, $(y - x)$ is the amount ordered.)

T - review period.

N - amount demanded during the review period $[0,T]$.

Note that N is an integer valued random variable having a contagious probability distribution

$$P_n(T) = \binom{\rho + n - 1}{n} \bar{p}^{\rho} \bar{q}^{n} \qquad \text{for all } n = 0,1,2, \ldots$$

where

(i) $\binom{\rho + n - 1}{n} = \dfrac{\Gamma(\rho + n)}{\Gamma(\rho)\Gamma(n + 1)}$

(ii) $\bar{p} = e^{-\alpha T}$, $\bar{p} \geqq 0$, $\bar{q} \geqq 0$ and $\bar{p} + \bar{q} = 1$. α is the unit contagion rate for the current period under consideration. $(\alpha > 0)$

(iii) $P_n(T)$ is the probability of n demands in $[0,T]$.

(iv) $\rho = \lambda'/\alpha$ and $\lambda' = \lambda + N'\alpha$. Where λ is the constant demand rate for the current period and N' is the total number of demands in the previous periods. Note for the first period $\lambda' \equiv \lambda$ since $N' \equiv 0$.

m(T) - mean number of demands in the current period. Note $m(T) = \rho\bar{q}/\bar{p}$.

k - set up cost. $(k \geqq 0)$

c - cost/unit purchased. $(c > 0)$

I - inventory holding cost per unit time per \$ invested in inventory.

h = Ic - inventory holding cost per unit time per unit held in inventory. $(h > 0)$

p - shortage cost/unit short. $(p \geqq 0)$

$\hat{p}$ - shortage cost/unit short/unit time. $(\hat{p} \geqq 0)$

r - gross revenue/unit sold. $(r > 0)$

2.3 Assumptions

A-I:

No disposal of goods is allowed at the end of the period. In other words, at the end of every period we either order for more to increase the inventory level $(\Rightarrow y > x)$ or order nothing and stay at the same initial inventory level $x (\Rightarrow y = x)$. Hence, $y \geq x$ always.

A-II:

The expected net revenue from unfilled orders are not included in the current period. Since the back orders are supplied only at the beginning of next period, the revenue from filling the back orders is included in the next period for convenience. This assumption is not correct if the current period is the last period since there is no succeeding period. This will be discussed in a later chapter. (Refer to Section 3.8.)

By Assumption A-I, we have to consider only two cases:

Case (i) $x \lesseqgtr 0$, $y \geq 0$, $y \geq x$

Case (ii) $x < 0$, $y < 0$, $y \geq x$.

The relevant costs in these two cases are discussed using Figures 2.1 and 2.2 for Case (i) and Case (ii) respectively.

2.4 Expected Gross Revenue Function

By Assumption A-II, the expected revenue corresponds to goods sold in the current period only.

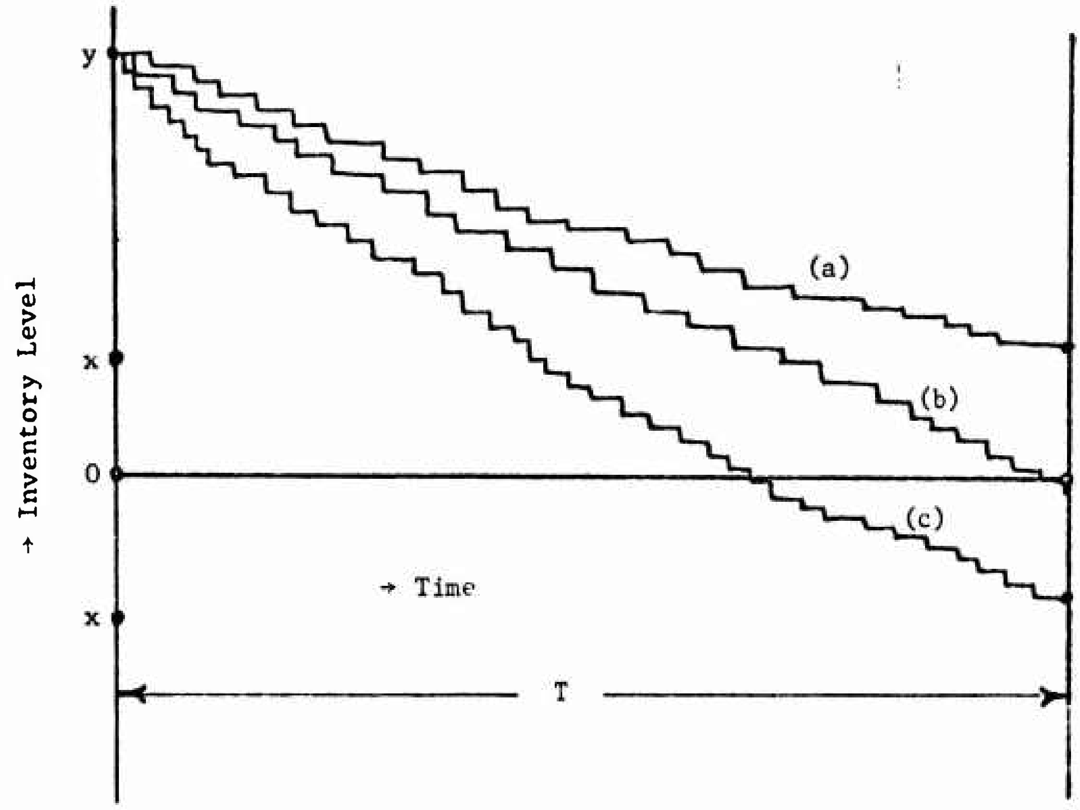

FIGURE 2.1

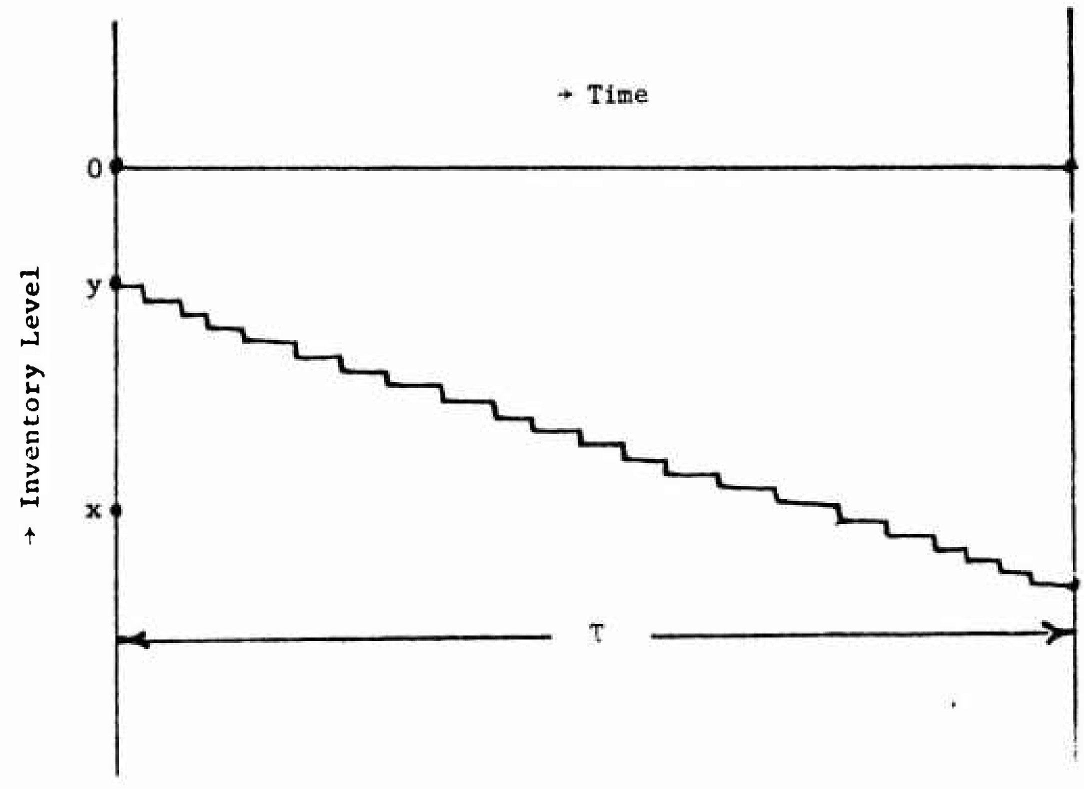

FIGURE 2.2

Case (i):

Referring to Figure 2.1, there is no revenue from filling last period's back order if x is nonnegative, while the revenue is $-rx$ if x is negative. Hence, the revenue from filling back orders is written as $-r \min(x,0)$ where x represents the algebraic value of the initial inventory level $\left(x \begin{smallmatrix} < \\ = \\ > \end{smallmatrix} 0\right)$. Similarly, the revenue from demands in the current period (Figure 2.1) is r times the number of demands for realizations (a) and (b) and is ry for (c). Hence, in general, the revenue from current demand is $r \sum_{n=0}^{y} nP_n(T) + r \sum_{n=y+1}^{\infty} yP_n(T)$. Thus, the total expected gross revenue is $-r \min(x,0) + r \sum_{n=0}^{y} nP_n(T) + r \sum_{n=y+1}^{\infty} yP_n(T)$. By changing the summation, this expression becomes $-r \min(x,0) + rm(T)$ $- r \sum_{n=y+1}^{\infty} (n - y)P_n(T)$.

Case (ii):

Referring to Figure 2.2 and Assumption A-II, the gross revenue is $r(y - x)$. (Note: $x < 0$ and $y < 0$.)

2.5 Procurement Cost Function

As stated in Section 1.13, this includes a set-up cost and the cost of purchasing the goods. This cost is the same for both Cases (i) and (ii). Since the set-up cost is positive only if an order is made, a delta function is introduced to take care of the case when no order is made (i.e., $y = x$). Hence, the set-up cost $= k\delta(y - x)$ where

$$\delta(y - x) = 1 \quad \text{if} \quad y > x$$
$$= 0 \quad \text{otherwise.}$$

The cost of goods purchased is $c(y - x)$. Hence, the procurement cost is $k\delta(y - x) + c(y - x)$.

2.6 Expected Inventory Holding Cost

Case (i):

An inventory holding cost is incurred as long as there is a positive inventory level and there is an instantaneous holding cost with respect to the inventory level at that instant. Considering an instant "t" ε [0,T] , the inventory holding cost in the interval [t,t + dt] will be $\sum_{n=0}^{y} (y - n)P_n(t)dt$, and integrating over the entire interval [0,T] , the expected inventory holding cost for the entire period is

$$h \sum_{n=0}^{y} (y - n) \int_0^T P_n(t)dt \ .$$

Now, the integral $\int_0^T P_n(t)dt$

$$= \int_0^T \frac{\Gamma(\rho + n)}{\Gamma(\rho)\Gamma(n + 1)} (e^{-\alpha t})^{\rho} (1 - e^{-\alpha t})^n dt \ .$$

By change of variables, with the substitution $u = 1 - e^{-\alpha t}$,

$$\int_0^T P_n(t)dt = \int_0^{\bar{q}} \frac{\Gamma(\rho + n)}{\Gamma(\rho)\Gamma(n + 1)} (1 - u)^{\rho-1} u^n \frac{du}{\alpha}$$

where

$$\bar{q} = 1 - e^{-\alpha T} \ .$$

The integrand is an Incomplete Beta Function Ratio (properties of incomplete Beta functions are given in Appendix II). By definition, a Beta function with parameters (m,n) is given by

$$B(m,n) = \int_0^1 u^{m-1}(1-u)^{n-1}du = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} ,$$

and the incomplete Beta function with parameters x , m and n is given by

$$B_x(m,n) = \int_0^x u^{m-1}(1-u)^{n-1}du \qquad \text{for } x < 1 .$$

Define

$$\frac{B_x(m,n)}{B(m,n)} = I_x(m,n) < 1 \qquad \text{for } x < 1$$

where $I_x(m,n)$ denotes the incomplete Beta function ratio.† Hence,

$$\int_0^T P_n(t)dt = \frac{1}{\alpha} \frac{I_{\bar{q}}(n+1,\rho)}{\rho + n} \tag{2.1}$$

and the expected inventory holding cost is $\frac{h}{\alpha} \sum_{n=0}^{y} (y-n) \frac{I_{\bar{q}}(n+1,\rho)}{\rho + n}$.

Case (ii):

It is immediate that the inventory holding cost in this case (Figure 2.2) is zero, as there is only a negative inventory level throughout the period.

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†Pearson [11] has tabulated the values of incomplete Beta functions for various parameters.

2.7 Expected Shortage Costs

As stated earlier, there are two kinds of shortage costs, one which depends on the amount of shortage and the other depending on the instantaneous shortage level. Again, Cases (i) and (ii) are discussed separately.

Case (i): (Figure 2.1)

The expected shortage cost associated with parameter "p" is nothing but the expected shortage level at the end of the period times the cost per unit short (p) . By computing the instantaneous shortage cost in a fashion similar to that for holding cost, the total expected shortage cost is

$$p \sum_{n=y+1}^{\infty} (n - y)P_n(T) + \frac{\hat{p}}{\alpha} \sum_{n=y+1}^{\infty} (n - y) \frac{I_{\underline{q}}(n + 1,\rho)}{\rho + n} .$$

Case (ii):

Referring to Figure (2.2), the total expected shortage cost is

$$-py + pm(T) - \hat{p}yT + \hat{p} \int_0^T m(t)dt$$

which reduces to

$$(p + \hat{p}/\alpha)m(T) - \hat{p}\rho T - (p + \hat{p}T)y ,$$

using the fact

$$m(t) = \frac{\rho(1 - e^{-\alpha t})}{e^{-\alpha t}} = \rho[e^{-\alpha t} - 1] .$$

2.8 Expected Net Revenue Function

The expected net revenue function for a given review period T with parameter y is denoted by $\pi(y,T)$. The expressions for $\pi(y,T)$ for Cases (i) and (ii) given separately are:

Case (i):

$$y \geq 0 , \; x \lesseqgtr 0 , \; y \geq x$$

$$\pi(y,T) = rm(T) - r\min(x,0) + cx - k\delta(y - x)$$

$$(2.2) \qquad - \left[cy + \frac{h}{\alpha} \sum_{n=0}^{y} (y - n) \frac{I_{\bar{q}}(n + 1,\rho)}{\rho + n} \right.$$

$$+ (p + r) \sum_{n=y+1}^{\infty} (n - y) P_n(T)$$

$$\left. + \hat{p}/\alpha \sum_{n=y+1}^{\infty} (n - y) \frac{I_{\bar{q}}(n + 1,\rho)}{\rho + n} \right] .$$

$P_n(T)$ and $I_{\bar{q}}$ can be computed from the tables available in [14] and [11] respectively.

Case (ii):

$$y < 0 , \; x < 0 , \; y \geq x$$

$$(2.3) \qquad \pi(y,T) = -rx + cx - k\delta(y - x)$$

$$- [m(T)(p + \hat{p}/\alpha) - \hat{p}\rho T - (p + \hat{p}T + r - c)y] .$$

It can be easily verified that the following general expression for both Cases (i) and (ii) holds. In general, for all $y \gtreqless 0$, $x \gtreqless 0$, $y \geq x$

$$(2.4) \qquad \pi(y,T) = rm(T) - r\min(x,0) + cx$$

$$- [k\delta(y - x) + G(y,T)]$$

where

$$G(y,T) = cy + (p + r) \sum_{n=0}^{\infty} [n - \min (y,n)]P_n(T)$$

$$(2.5) \qquad + \frac{h}{\alpha} \sum_{n=0}^{\infty} [\max (y,n) - n] \frac{I_{\underline{q}}(n + 1,\rho)}{\rho + n}$$

$$+ \hat{p}/\alpha \sum_{n=0}^{\infty} [n - \min (y,n)] \frac{I_{\underline{q}}(n + 1,\rho)}{\rho + n} .$$

2.9 Properties of the Cost Function

The optimal order level y, that maximizes the net profit $\pi(y,T)$, is obtained by minimizing the total cost function $k\delta(y - x) + G(y,T)$ where $k\delta(y - x)$ is a step function as shown below in Figure 2.3.

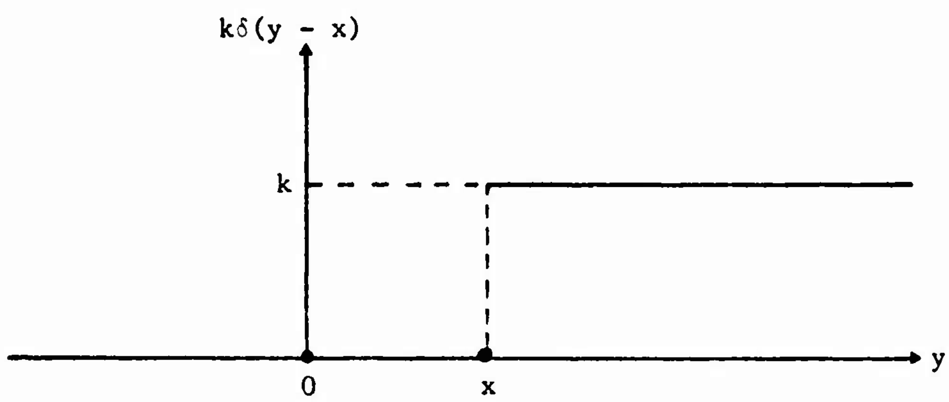


FIGURE 2.3

Hence, the main interest is to examine the properties of the function $G(y,T)$ for all values of y.

Proposition 2.1:

The cost function $G(y,T)$ is strictly pointwise convex in $y \in \{0,1,2, \ldots, \infty\}$, for given $T \geq 0$.

Proof:

Substituting $y \geqq 0$ in (2.5),

$$G(y,T) = cy + (p + r) \sum_{n=y+1}^{\infty} (n - y)P_n(T)$$

$$(2.6) \qquad + \frac{h}{\alpha} \sum_{n=0}^{y} (y - n) \frac{I_{\bar{q}}(n + 1,\rho)}{\rho + n}$$

$$+ \hat{p}/\alpha \sum_{n=y+1}^{\infty} (n - y) \frac{I_{\bar{q}}(n + 1,\rho)}{\rho + n} .$$

To show that $G(y,T)$ is strictly pointwise convex in y, it is sufficient to show that its second difference (as y takes integer values only) is positive. The first difference of $G(y,T)$ is

$$\Delta G(y,T) \triangleq G(y + 1,T) - G(y,T)$$

$$(2.7) \qquad = -(p + r - c) + (p + r) \sum_{n=0}^{y} P_n(T)$$

$$+ \frac{h}{\alpha} \sum_{n=0}^{y} \frac{I_{\bar{q}}(n + 1,\rho)}{\rho + n} - \hat{p}/\alpha \sum_{n=y+1}^{\infty} \frac{I_{\bar{q}}(n + 1,\rho)}{\rho + n} .$$

The second difference of $G(y,T)$ is

$$\Delta^2 G(y,T) = \Delta G(y + 1,T) - \Delta G(y,T) = (p + r)P_{y+1}(T)$$

$$+ \left(\frac{h + \hat{p}}{\alpha}\right) \frac{I_{\bar{q}}(y + 2,\rho)}{\rho + y + 1} .$$

We have $P_{y+1}(T) > 0$ and $I_{\bar{q}}(y + 2,\rho) > 0$ for all $y \in \{0,1,2, \ldots, \infty\}$ and $T \geqq 0$. Hence $\Delta^2 G(y,T) > 0$, implying that $G(y,T)$ is strictly pointwise convex in $y \in \{0,1,2, \ldots, \infty)$, for given $T \geqq 0$.

Proposition 2.2:

The values of the cost function $G(y,T)\big|_{y=0^-}$ and $G(y,T)\big|_{y=0^+}$ (which represents the values of $G(0,T)$ reached from left and right of the origin respectively) are equal.

Proof:

Putting $y \leq 0$ in (2.5),

$$(2.8)\quad G(y,T) = (p + r + \hat{p}/\alpha)m(T) - \hat{p}\rho T - (p + \hat{p}T + r - c)y \ .$$

Hence, $G(y,T)$ is linear for all $y < 0$ with a negative slope since $r > c$. Its value at the origin is

$$(2.9)\qquad G(y,T)\big|_{y=0^-} = (p + r + \hat{p}/\alpha)m(T) - \hat{p}\rho T \ .$$

From (2.6),

$$(2.10)\qquad G(y,T)\big|_{y=0^+} = (p + r + \hat{p}/\alpha)m(T) - \hat{p}\rho T \ .$$

By (2.9) and (2.10), the values of $G(y,T)$ at the origin from the left and from the right are the same.

Proposition 2.3:

$G(y,T)$ is pointwise convex in y, for given $T \geq 0$.

Proof:

From (2.8), $G(y,T)$ is linear for all $y \leq 0$ and hence pointwise convex in $y \leq 0$. From Proposition 2.1, $G(y,T)$ is strictly pointwise convex for all $y \in \{0,1,2, \ldots, \infty\}$. Hence, the only remaining thing to be shown is

$$\Delta G(y,T)\Big|_{y=0^+} \geqq \Delta G(y,T)\Big|_{y=0^-} .$$

From (2.8),

$$\Delta G(y,T)\Big|_{y=0^-} = -(p + \hat{p}T + r - c) .$$

From (2.7), putting $y = 0$, we get

$$\Delta G(y,T)\Big|_{y=0^+} = -(p + r - c) + (p + r)P_o(T)$$

$$+ \frac{h}{\alpha}\frac{I_{\bar{q}}(1,\rho)}{\rho} - \hat{p}/\alpha \sum_{n=1}^{\infty} \frac{I_{\bar{q}}(n + 1,\rho)}{\rho + n}$$

using (2.1) that $\int_0^T P_n(t)dt = \frac{1}{\alpha}\frac{I_{\bar{q}}(n + 1,\rho)}{\rho + n}$, we have

$$\Delta G(y,T)\Big|_{y=0^+} = -(p + r - c) + (p + r)P_o(T)$$

$$+ h\int_0^T P_o(t)dt - \hat{p}\int_0^T [1 - P_o(t)]dt$$

(2.11) $$= -(p + \hat{p}T + r - c) + (p + r)P_o(T)$$

$$+ (h + \hat{p})\int_0^T P_o(t)dt > \Delta G(y,T)\Big|_{y=0^-} .$$

Hence, $G(y,T)$ is pointwise convex throughout in y for all $T \geqq 0$.

2.10 Critical Order Level

Let $y_o(T)$ denote the critical order level for given T minimizing the cost function $G(y,T)$. Since $G(y,T)$ is linear with negative slope for $y \leqq 0$, it is immediate that the global minimum of $G(y,T)$ cannot occur for $y < 0$. Since $G(y,T)$ is strictly pointwise convex for y in nonnegative real line, $y_o(T)$ is unique. If the slope (first difference) of $G(y,T)$ at $y = 0^+$ is positive, then the minimum of $G(y,T)$ occurs at $y = 0$. If, on the other hand, $\Delta G(y,T)$ at $y = 0^+$ is nonpositive, then $y_o(T) > 0$. Hence, it is necessary to consider the sign of $\Delta G(y,T)$ at $y = 0^+$. Let

$$a(T) = \Delta G(y,T)\Big|_{y=0^+} .$$

From (2.11)

$$a(T) = -(p + r + \hat{p}T - c) + (p + r)P_o(T) + (h + \hat{p}) \int_0^T P_o(t)dt ,$$

and since $P_o(t) = e^{-\lambda t}$, it follows that

$$\text{(2.12)} \quad a(T) = -(p + r + \hat{p}T - c) + \frac{h + \hat{p}}{\lambda} + \left(p + r - \frac{h + \hat{p}}{\lambda}\right)P_o(T) .$$

Thus, if $a(T) > 0$ for given T, then $y_o(T) \equiv 0$. Otherwise, $y_o(T) > 0$ for $a(T) \leq 0$. As the review period T tends to zero, $a(T)$ tends to a positive value c. Also, $a(T)$ tends asymptotically to $-(p + r - c) + \frac{h + \hat{p}}{\lambda} - \hat{p}T$ for $T \to \infty$, which is negative and remains so for sufficiently large values of T.

<u>Proposition 2.4</u>:

There exists a unique and finite $T_o > 0$ such that $a(T_o) = 0$ and $a(T) \gtrless 0$ for $T \lessgtr T_o$.

<u>Proof</u>:

From (2.12),

$$a'(T) = \frac{da(T)}{dT} = \left[\frac{h + \hat{p}}{\lambda} - (p + r)\right] \lambda P_o(T) - \hat{p}$$

and

$$a''(T) = -\left[\frac{h + \hat{p}}{\lambda} - (p + r)\right] \lambda^2 P_o(T) \; .$$

Consider the two cases where

(a) $\frac{h + \hat{p}}{\lambda} \leqq (p + r)$

(b) $\frac{h + \hat{p}}{\lambda} > (p + r)$.

Case (a):

Since $\frac{h + \hat{p}}{\lambda} \leqq (p + r)$, $a'(T)$ is strictly negative and $a(T)$ is a convex decreasing function in T . Hence, there exists a unique T_o such that

$$a(T) \gtreqless 0 \quad \text{for} \quad T \lesseqgtr T_o \; .$$

Case (b):

Since $\frac{h + \hat{p}}{\lambda} > (p + r)$, $a''(T)$ is negative. Hence, $a(T)$ is strictly concave. Thus, $a'(T)$ decreases as T increases and becomes negative after some T and stays negative thereafter. Hence, there exists a unique and finite T_o where $a(T_o) = 0$ and $a(T) \gtrless 0$ for $T \lessgtr T_o$.

From Proposition 2.4, it follows immediately that

$$(2.13) \qquad \begin{aligned} y_o(T) &\equiv 0 && \text{for all } 0 \leqq T < T_o \\ &> 0 && \text{for all } T \geqq T_o \; . \end{aligned}$$

For $T \geqq T_o$, the critical order level (integer) $y_o(T)$ is sought such that

$$\begin{aligned} \Delta G(y_o(T) - 1,T) &\leqq 0 \\ \Delta G(y_o(T),T) &> 0 \ . \end{aligned} \tag{2.14}$$

A plot of the function $G(y,T)$ with respect to y , for a given T , is shown in Figure 2.4.

2.11 Optimal Order Level

Given an initial order level x , if the order level is $y_o(T)$, (assuming $y_o(T) > x$), then the total cost of ordering and operating at level $y_o(T)$ is $k + G(y_o(T),T)$. Instead, if the order level remains at x , the cost is $G(x,T)$. Hence, the optimal policy would be to order up to $y_o(T)$ if and only if, $G(x,T) > k + G(y_o(T),T)$. Otherwise, no additional stock is ordered. Compute a level $s(T) < y_o(T)$, such that

$$G(s(T),T) = k + G(y_o(T),T) \ . \tag{2.15}$$

Note that $s(T)$ may be positive or negative for given T .

Hence, the optimal order level, denoted by $y^*(T)$, will be

$$y^*(T) = \begin{cases} x & \text{if } x \geqq s(T) \\ y_o(T) & \text{if } x < s(T) \ . \end{cases} \tag{2.16}$$

Denoting $y_o(T)$ by S , such an order policy is known as an $(s - S)$ policy, since the order level is S if the inventory level is below s ; otherwise, it is optimal to remain at the level x .

2.12 Algorithm to Find the Optimal Order Policy

For a given review period T , the optimal order level is found from

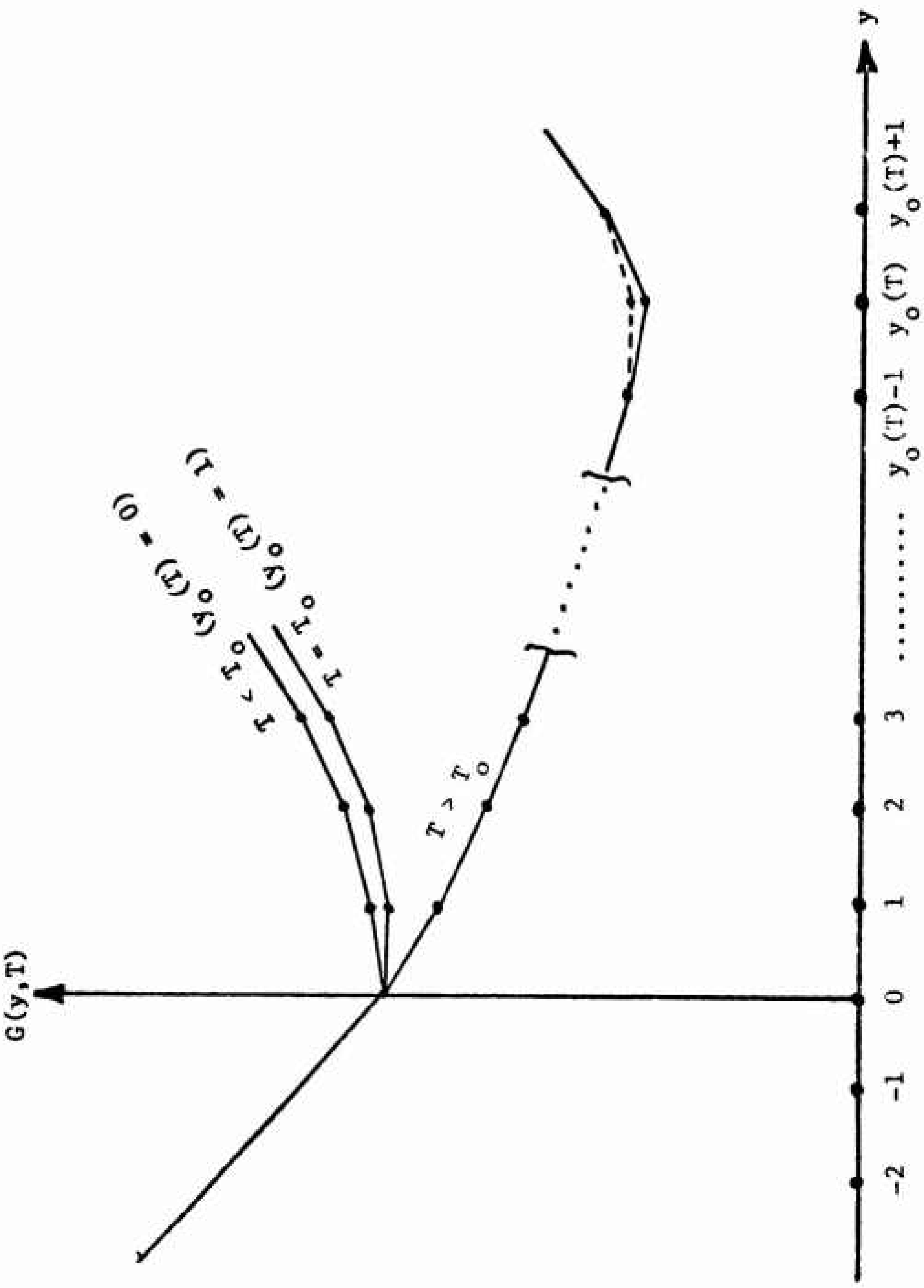


FIGURE 2.4

$y_o(T)$ by using (2.15) and (2.16). Hence, the main interest is to have an algorithm to find $y_o(T)$, for those periods T which are greater than or equal to T_o, since $y_o(T)$ is zero for all values of T which are less than T_o. From (2.14), an integer $y_o(T)$ is sought such that

$$\Delta G(y_o(T) - 1, T) \leqq 0$$
$$\Delta G(y_o(T), T) > 0 .$$

Assume there are two nonnegative levels y_1 and $y_2 (y_2 > y_1)$ such that

$$\Delta G(y_1, T) \leqq 0 \quad \text{and} \quad \Delta G(y_2, T) > 0 .$$

Note that y_2 is the critical order level for given T, if $y_2 = y_1 + 1$. At the start of the algorithm the interval $[y_1, y_2]$ will be much larger than unity. The aim is to reduce the interval length such that ultimately $[y_2 - y_1]$ equals unity. For this purpose, the well-known bisection procedure may be used. Later a procedure for finding the initial values of y_1 and y_2 will be given.

Bisection Procedure:

(i) Compute the median, $y_m = \dfrac{y_1 + y_2}{2}$.

(ii) Calculate $\Delta G(y_m, T)$ using (2.7).

(iii) If $\Delta G(y_m, T) > 0$, then set $y_2 = y_m$. Otherwise, set $y_1 = y_m$. Now we have a new and shorter interval $[y_1, y_2]$.

(iv) If the length of the interval $(y_2 - y_1)$ is unity, then set $y_2 = y_o(T)$. Otherwise, go back to Step (i).

Computation of the Initial Values of y_1 and y_2:

Since $T > T_o$, we know from Theorem 2.4, that $\Delta G(y,T)\big|_{y=0^+} < 0$.

Hence, we set $y_1 = 0$. Referring to Equation (2.7), it is possible to find an M , with the help of the incomplete Beta function tables [11], such that $I_{\bar{q}}(M + 1,\rho) \to 0$. In which case, $\Delta G(M,T) > 0$ as required. For example, for $\rho = 2$, $I_{.3} \to 0$ as $M \to 8$; $I_{.8} \to 0$ as $M \to 50$. Hence, we set $y_2 = M$.

2.13 Properties of the Critical Order Level

It will be of interest and use (refer to Chapter 3) if we know the properties of $y_o(T)$ as the review period varies in the nonnegative real line. The following properties of $y_o(T)$ with respect to T will be proved with the help of a few propositions:

(i) $y_o(T)$ is nondecreasing in $T \in [0,\infty)$.

(ii) $y_o(T)$ is a step function of $T \in [0,\infty)$.

(iii) $y_o(T)$ is bounded above as $T \to \infty$.

Define a function $F_N(T)$ such that

$$\begin{aligned} F_N(T) = (p + r) \sum_{n=0}^{N} P_n(T) + \frac{h}{\alpha} \sum_{n=0}^{N} \frac{I_{\bar{q}}(n + 1,\rho)}{\rho + n} \\ - \hat{p}/\alpha \sum_{n=N+1}^{\infty} \frac{I_{\bar{q}}(n + 1,\rho)}{\rho + n} \qquad \text{for all } N = 0,1,2, \ldots \end{aligned} \tag{2.17}$$

Note that $F_N(T)$ equals $\Delta G(N,T) + (p + r - c)$. From (2.14), N is the critical order level if and only if

$$\begin{aligned} F_{N-1}(T) &\leqq (p + r - c) \\ F_N(T) &> (p + r - c) . \end{aligned} \tag{2.18}$$

Hence, for a given T , an integer N is sought such that $F_N(T)$ is above the "(p + r - c) line[+]" and $F_{N-1}(T)$ is just below the "(p + r - c) line."

Proposition 2.5:

$F_{N+1}(T)$ is greater than $F_N(T)$ for all $T > 0$ and for all $N = 0,1,2, \ldots$.

Proof:

It is sufficient to show that the first difference of $F_N(T)$ is positive. Calculate

$$\Delta F_N(T) = F_{N+1}(T) - F_N(T)$$

$$= (p + r)P_{N+1}(T) + \left(\frac{h + \hat{p}}{\alpha}\right) \frac{I_{\bar{q}}(N + 2,\rho)}{\rho + N + 1}$$

$$> 0 \quad \text{for all} \quad T > 0 \; .$$

Proposition 2.5 shows that $y_o(T)$ is unique for every T and it is a step function of T . The next thing to show is the existence of finite upper bound to $y_o(T)$. For this, the derivative of $F_N(T)$ is needed which involves the derivatives of $P_n(T)$ and $I_{\bar{q}}(n + 1,\rho)$. Now, from (1.10),

$$P'_n(T) = \frac{d}{dT}[P_n(T)] = \frac{d}{dT}\left[\binom{\rho + n - 1}{n}(e^{-\alpha T})^{\rho}(1 - e^{-\alpha T})^n\right]$$

(2.19)

$$= (\rho + n - 1)\bar{p}\alpha P_{n-1}(T) - \lambda P_n(T) \qquad \text{for all} \quad n = 0,1,2, \ldots$$

Note:

$$P_{-1}(T) \triangleq 0 \; , \; \bar{p} \triangleq e^{-\alpha T} \; , \; \bar{q} \triangleq 1 - e^{-\alpha T} \; \text{and} \; \bar{p} + \bar{q} \triangleq 1 \; .$$

[+] In a graph of $F_N(T)$ against T , "(p + r - c) line" is the line drawn parallel to T-axis (abscissa) at a height of $(p + r - c)$.

From (2.1),

$$\frac{d}{dT}\left[\frac{1}{\alpha}\,\frac{I_{\bar{q}}(n+1,\rho)}{\rho+n}\right] = P_n(T) .$$

Hence, we can write

$$F_N'(T) = \frac{dF_N(T)}{dT}$$

(2.20) $$= (p + r) \sum_{n=0}^{N} [(\rho + n - 1)\bar{p}\alpha P_{n-1}(T) - \lambda P_n(T)]$$

$$+ (h + \hat{p}) \sum_{n=0}^{N} P_n(T) - \hat{p} \qquad \text{for all } N \geq 1 .$$

To treat the case $N = 0$ separately, it has been omitted in (2.20). From (2.17), putting $N = 0$ and simplifying,

(2.21) $$F_o(T) = -\hat{p}T + (p + r)P_o(T) + \frac{h + \hat{p}}{\lambda}[1 - P_o(T)]$$

(2.22) $$F_o'(T) = -\hat{p} - [\lambda(p + r) - (h + \hat{p})]P_o(T) .$$

Limiting Values

Next is to examine the limiting values of $F_N(T)$ and $F_N'(T)$ for all $N = 0,1,2, \ldots$ as $T \to 0$ and $T \to \infty$.

(i) $T \to 0$:

$$F_N(T) \to (p + r) \qquad \text{for all } N = 0,1,2, \ldots$$

(2.23) $$F_o'(T) \to -[\lambda(p + r) - h]$$

$$F_N'(T) \to h \qquad \text{for } N \geq 1 .$$

(ii) As $T \to \infty$:

$$F_N(T) \to -\infty \qquad \text{for all finite } N = 0,1,2,\ldots$$

$$\text{and for } \hat{p} > 0$$

$$F_N'(T) \to -\hat{p} \qquad \text{for all finite } N = 0,1,2,\ldots$$

The following proposition characterizes the behavior of $F_N(T)$ and $F_N'(T)$ for infinitely large N .

Proposition 2.6:

The critical order level is bounded above as the review period $T \to \infty$.

Proof:

From (2.20), as N tends to ∞ ,

$$F_N'(T) \to (p + r) \sum_{n=0}^{\infty} [(\rho + n - 1)\bar{p}\alpha P_{n-1}(T) - \lambda P_n(T)] + (h + \hat{p}) - \hat{p}$$

$$= (p + r)\bar{p}\alpha\left[\rho + \frac{\rho\bar{q}}{\bar{p}}\right] - \lambda(p + r) + h = h \qquad \text{for all } T \geq 0 .$$

Hence, for infinitely large N , $F_N(T)$ is linear with positive slope, and from (2.23) $F_N(T) > (p + r - c)$ for all values of $T \geq 0$ and for N greater than some bounded integer. Hence, condition (2.18) cannot be satisfied by an infinitely large N which implies that $y_o(T)$ is finite for all values of T .

To show the most important and difficult property that $y_o(T)$ is a nondecreasing function of T , it has to be proved that $F_N(T)$ crosses the "$(p + r - c)$ line" at most once for all finite N . Through the following proposition, a much stronger result is proved.

Proposition 2.7:

$F_N''(T)$ can have at most one change in sign in the range $T \in [0,\infty)$ for all finite $N = 0,1,2, \ldots$.

Proof:

Consider first $N \geq 1$. Rewrite (2.20) as

$$F_N'(T) = (p + r) \sum_{n=1}^{N} [(\rho + n - 1)\bar{p}\alpha P_{n-1}(T)]$$

$$- [\lambda(p + r) - (h + \hat{p})] \sum_{n=0}^{N} P_n(T) - \hat{p} \qquad \text{for all } N \geq 1 .$$

Introducing a new variable $u = \bar{q} = 1 - e^{-\alpha T}$, write $F_N'(T)$ as a function of "u" alone. Note as T varies in the range $[0,\infty)$, u varies in the half open interval $[0,1)$ and the transformations from u to T and vice versa are one to one. Hence, $P_n(T)$ reduces to

$$P_n(u) = c_n(1 - u)^{\rho} u^n \quad \text{where} \quad c_n = \frac{\Gamma(\rho + n)}{\Gamma(\rho)\Gamma(n + 1)}$$

and $F_N'(T)$ is transformed to

$$F_N'(u) = \left\{ \alpha(p + r) \sum_{n=1}^{N} (\rho + n - 1)c_{n-1}(1 - u)^{\rho+1}u^{n-1} \right.$$

$$\left. - [\lambda(p + r) - (h + \hat{p})] \sum_{n=0}^{N} c_n(1 - u)^{\rho}u^n \right\} - \hat{p} .$$

Taking the term $(1 - u)^{\rho}$ outside the curly brackets and noting that $(\rho + n - 1)c_{n-1} = nc_n$,

$$F_N'(u) = (1-u)^\rho \left\{ \alpha(p+r)\rho \sum_{n=0}^{N-1} c_n u^n + \alpha(p+r) \sum_{n=0}^{N-1} nc_n u^n \right.$$

$$\left. - \alpha(p+r) \sum_{n=0}^{N} nc_n u^n - [\lambda(p+r) - (h+\hat{p})] \sum_{n=0}^{N} c_n u^n \right\} - \hat{p}$$

cancelling the like terms, this simplifies to

$$(2.24) \qquad F_N'(u) = (1-u)^\rho \left\{ (h+\hat{p}) \sum_{n=0}^{N} c_n u^n - (\lambda + \alpha N)(p+r) c_N u^N \right\} - \hat{p}$$

$$\text{for all} \quad N \geq 1 \ .$$

For a check note that

$$F_N'(0) = h \quad \text{and} \quad F_N'(1) = -\hat{p} \ .$$

Since

$$(2.25) \qquad F_N''(T) = F_N''(u) \frac{du}{dT} = F_N''(u)\alpha(1-u) \ ,$$

$\alpha > 0$ and $(1-u) > 0$ for all $u \in [0,1)$, the sign of $F_N''(T)$ is the same as that of $F_N''(u)$. From (2.24),

$$F_N''(u) = (1-u)^\rho \left[(h+\hat{p}) \sum_{n=1}^{N} nc_n u^{n-1} - (\lambda+\alpha N)(p+r) N c_N u^{N-1} \right]$$

$$- \rho(1-u)^{\rho-1} \left[(h+\hat{p}) \sum_{n=0}^{N} c_n u^n - (\lambda+\alpha N)(p+r) c_N u^N \right] .$$

Taking $(1-u)^{\rho-1}$ as a common factor,

$$F_N''(u) = (1-u)^{\rho-1}\left\{\left[(h+\hat{p})\sum_{n=1}^{N}(\rho+n-1)c_{n-1}u^{n-1}(1-u)\right]\right.$$

$$-\left[(\lambda+\alpha N)(p+r)Nc_N u^{N-1}(1-u)\right]$$

$$\left.-\left[\rho(h+\hat{p})\sum_{n=0}^{N}c_n u^n\right]+\left[\rho(\lambda+\alpha N)(p+r)c_N u^N\right]\right\}$$

$$= (1-u)^{\rho-1}\left\{(h+\hat{p})\left[\sum_{n=0}^{N-1}(\rho+n)c_n u^n - \sum_{n=1}^{N}(\rho+n-1)c_{n-1}u^n\right]\right.$$

$$-(\lambda+\alpha N)(p+r)Nc_N u^{N-1}+(\lambda+\alpha N)(p+r)Nc_N u^N$$

$$\left.-\rho(h+\hat{p})\sum_{n=0}^{N}c_n u^n+\rho(\lambda+\alpha N)(p+r)c_N u^N\right\}$$

using the fact that $(\rho+n-1)c_{n-1} \equiv nc_n$ and after cancellation of like terms,

$$F_N''(u) = (1-u)^{\rho-1}\left\{-(h+\hat{p})Nc_N u^N - (\lambda+\alpha N)(p+r)Nc_N u^{N-1}\right.$$

$$\left.+(\lambda+\alpha N)(p+r)Nc_N u^N - \rho(h+\hat{p})c_N u^N + \rho(\lambda+\alpha N)(p+r)c_N u^N\right\}.$$

Hence,

$$\begin{aligned}F_N''(u) &= c_N(1-u)^{\rho-1}u^{N-1}\{[(\lambda+\alpha N)(p+r)-(h+\hat{p})](N+\rho)u \\ &\quad -(\lambda+\alpha N)(p+r)N\} \qquad \text{for all } N \geq 1 .\end{aligned} \tag{2.26}$$

From (2.26), the sign of $F_N''(u)$ or equivalently the sign of $F_N''(T)$ depends only on that of the terms within the curly brackets, since $(1-u)^{\rho-1}$, c_N, u^{N-1} are all positive for $u \in [0,1)$. The sign of the terms within curly brackets in (2.26), depends on the values of the parameters and there are only two possible cases:

(i)

$$(\lambda + \alpha N)(p + r) > (h + \hat{p}) .$$

(True in practical cases.) In this case, the terms within the curly brackets can have at most one change in sign as they are linear in u . Hence, $F_N''(T)$ can have at most one change in sign in the range $T \in [0,\infty)$.

(ii)

$$(\lambda + \alpha N)(p + r) \leqq (h + \hat{p}) .$$

In this case, $F_N''(u) < 0$ for all $u \in (0,1) \Rightarrow F_N''(T) < 0$ for all $T \in (0,\infty)$. Hence, $F_N''(T)$ does not have any change in sign in the range $T \in [0,\infty)$ and for all $N \geqq 1$.

Proposition 2.7 is proved for all $N \geqq 1$. For the case when $N = 0$, from (2.22),

$$F_o''(T) = [\lambda(p + r) - (h + \hat{p})]\lambda P_o(T) . \tag{2.27}$$

This implies $F_o''(T)$ is either positive or nonpositive depending upon whether $\lambda(p + r) - (h + \hat{p}) > 0$ or $\lambda(p + r) - (h + \hat{p}) \leqq 0$ respectively. Hence, $F_o''(T)$ cannot have any change in sign.

There are three important corollaries which follow from Proposition 2.7 which ultimately prove that $y_o(T)$ is a nondecreasing function of T .

<u>Corollary 1</u>:

(a) For values of $N \geqq 1$, $F_N(T)$ is either strictly concave throughout or initially strictly concave and then remains strictly convex in T in the range $[0,\infty)$.

(b) $F_o(T)$ is either strictly concave throughout or strictly convex throughout in $T \in [0,\infty)$.

Proof:

Again consider the two cases relative to the magnitude of the parameters.

(i)

$$(\lambda + \alpha N)(p + r) > (h + \hat{p}) .$$

By (2.27), $F_o''(T) > 0$. Hence, $F_o(T)$ is strictly convex in T . By Proposition 2.7, $F_N''(T)$ can have at most one change in sign for all $N \geq 1$. From (2.26), note that $F_N''(T)$ is initially negative. Hence, $F_N(T)$ is either strictly concave throughout or strictly concave at first and then strictly convex throughout as T ranges $[0,\infty)$.

(ii)

$$(\lambda + \alpha N)(p + r) \leq (h + \hat{p}) .$$

By (2.27), $F_o''(T) < 0$. By (2.26), $F_N''(T) < 0$. Hence, $F_N(T)$ is strictly concave in $T \in [0,\infty)$ for $N = 0,1,2, \ldots$.

Corollary 2:

For all values of $N \in \{0,1,2, \ldots\}$, $F_N(T)$ crosses the "$p + r - c$ line" at most once and from above.

Proof:

Since $F_N(0) = p + r$ for all $N = 0,1,2, \ldots$, $F_N(T)$ starts from above the "$(p + r - c)$ line." Also, $F_N'(0) = h$ for all $N = 1,2, \ldots$ and $F_N'(\infty) = -\hat{p}$ for finite $N \geq 0$. Hence, from Proposition 2.7 and Corollary 1, it is clear that once $F'_N(T)$ becomes negative, it stays negative thereafter. Hence, Corollary 2 is true. For those N which are greater than some bounded integer, $F_N(T)$ increases continuously and hence will never cross the "p + r - c line."

Corollary 3:

The solutions $T_0, T_1, \ldots, T_{\bar{N}-1}$ (where $\bar{N}$ is the limiting value of $y_o(T)$) obtained by equating $F_N(T) = p + r - c$ for $N = 0,1,2, \ldots, (\bar{N}-1)$ respectively are unique and are strictly increasing.

Proof:

Using (2.18) and Corollary 2 and the fact that $\bar{N}$ is the limiting value of $y_o(T)$, it is immediate that $F_N(T)$ crosses the "(p + r - c) line" only once for $N = 0,1,2, \ldots, (\bar{N}-1)$. Hence, the solutions $T_0, T_1, \ldots, T_{\bar{N}-1}$ are unique. By Proposition 2.5, it is immediate that $T_0 < T_1 < T_2 < \ldots < T_{\bar{N}-1}$.

Example:

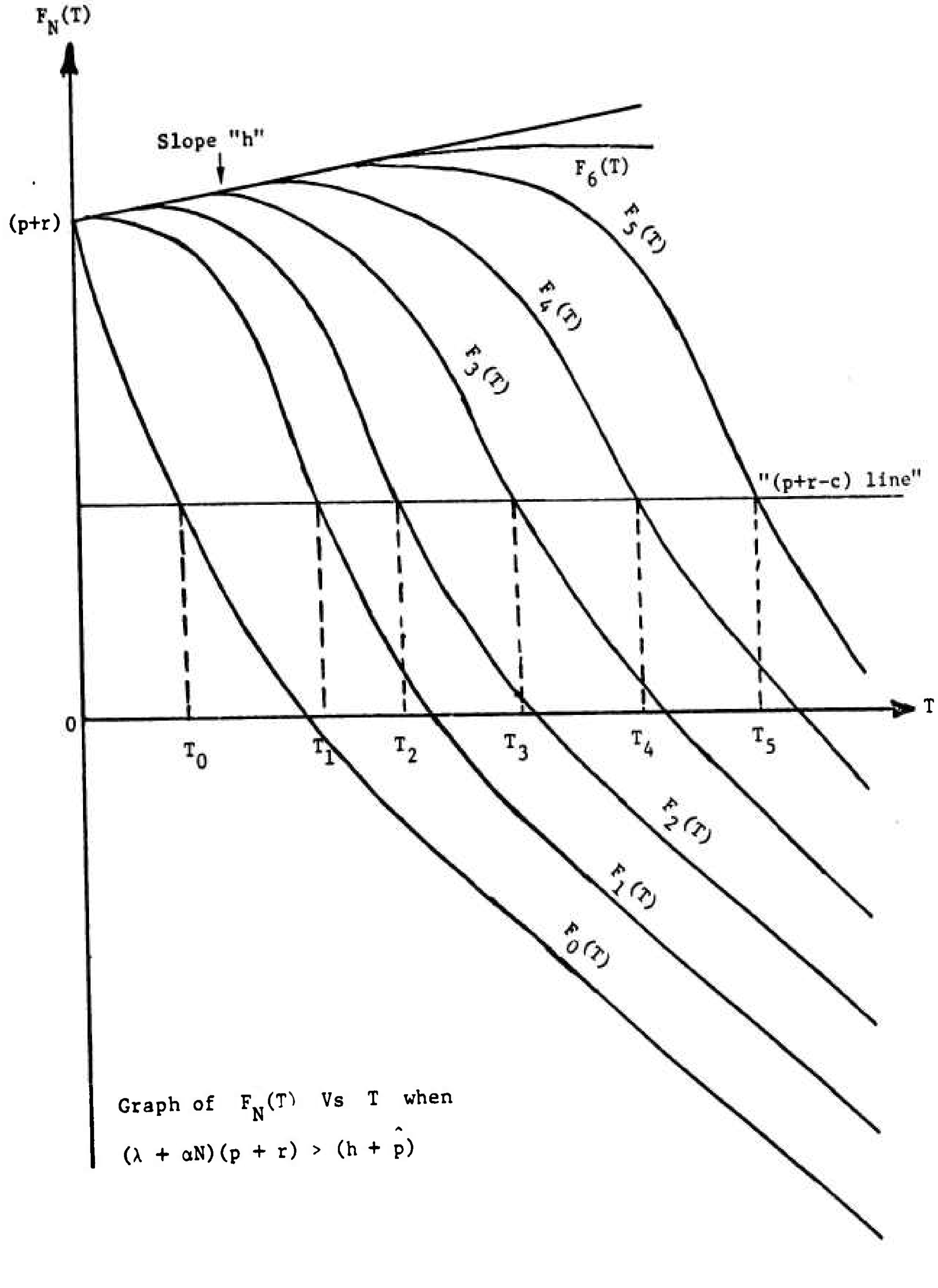
Figure 2.5 illustrates the shape of $F_N(T)$ with respect to $T \in [0,\infty)$, for the case when $(\lambda + \alpha N)(p + r) > (h + \hat{p})$, for values of $N = 0,1,2, \ldots, 6$.

Proposition 2.8:

The critical order level is a nondecreasing step function of the review period, continuous from the right with finite saltus at points $T_0, T_1, \ldots, T_{\bar{N}-1}$.

Proof:

From Corollary 2 and Proposition 2.5, it follows for a given $T = \hat{T}$,



Graph of $F_N(T)$ Vs T when

$(\lambda + \alpha N)(p + r) > (h + \hat{p})$

FIGURE 2.5

that if the critical order level is $y_o(\dot{T})$, then for all $T > \dot{T}$, condition (2.18) can only be satisfied by an $N \geq y_o(\dot{T})$. Hence, $y_o(T) \geq y_o(\dot{T})$ for all $T > \dot{T}$. The rest of the proposition follows from Corollary 3 and Equation (2.18). Thus,

$$
\begin{array}{ll}
y_o(T) = 0 & \text{for } 0 \leq T < T_0 \\
y_o(T) = 1 & \text{for } T_0 \leq T < T_1 \\
\vdots & \\
y_o(T) = \bar{N} - 1 & \text{for } T_{\bar{N}-2} \leq T < T_{\bar{N}-1} \\
y_o(T) = \bar{N} & \text{for } T \geq T_{\bar{N}-1} .
\end{array}
$$

<u>Example</u>:

Figure (2.6) illustrates the plot of $y_o(T)$ against T for the example given in Figure 2.5.

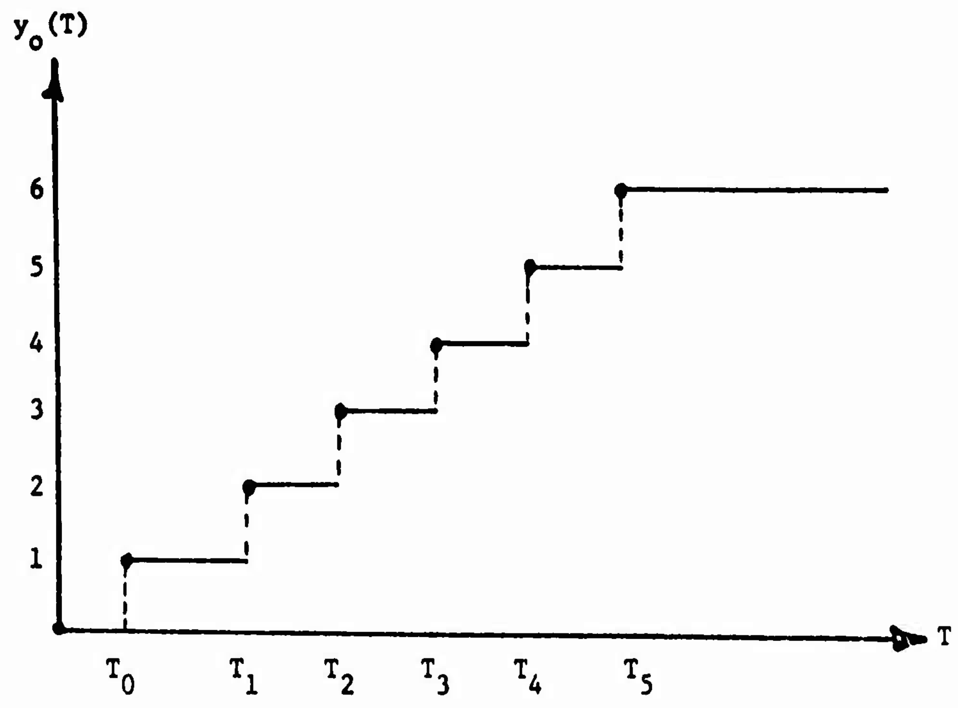


FIGURE 2.6

CHAPTER 3

SINGLE PERIOD MODEL FOR SEASONAL OR STYLE GOODS

3.0 Introduction

As discussed in Section 1.11, for the case of seasonal or style goods, either a fixed period-length policy or an optimal period-length policy may be followed. In the former case, an optimal order policy is sought by fixing the review period institutionally, while in the latter case an optimal order policy as well as an optimal review period are sought which leads to an optimization problem with two variables. The optimization is carried out in two stages by first finding the optimal ordering policy as a function of the length of the review period and then selecting an optimal review period. Thus, an optimal solution for both fixed and optimal period-length policies are given. Only a single period problem is solved for seasonal goods since in general the season is not longer than a year.

Here again two situations may arise from demands which occur when the inventory is zero:

(i) The demand is not filled, i.e., it is lost. The cost of losing these demands may be estimated by the stock-up cost parameters $p(\geq 0)$ and $\hat{p}(\geq 0)$ as explained in Section 1.15. If the cost of "lost sales" is ignored, then both p and $\hat{p}$ will be zero.

(ii) The demand is back ordered and supplied at the end of the season. In general, it is not practical to follow this policy for seasonal goods.

3.1 Optimal Order Level for a Given Review Period

Confining the analysis to the case when "lost sales" policy is followed, it is immediate that all the results of Chapter 2 hold by making the initial order level (x) zero. Thus the critical order level $y_o(T)$ satisfies,

$$\Delta G(y_o(T) - 1,T) \leqq 0$$
$$\Delta G(y_o(T),T) > 0$$

where $\Delta G(y,T)$ is given by (2.7). The optimum order level $y^*(T)$ is,

$$y^*(.) = \begin{cases} y_o(T) & \text{if } k + G(y_o(T),T) < G(0,T) \\ 0 & \text{otherwise.} \end{cases}$$

Since a single period problem is solved for seasonal goods, $y^*(T)$ equals zero implies no business is done. Hence, to be in business, a positive inventory level must be maintained at the beginning. For this to be true, the condition $k + G(y_o(T),T) < G(0,T)$ must be satisfied. Proposition 3.7 (refer to Section 3.5) proves that there exists a lower bound T_ℓ(> T_0 and finite) on the review period T, above which $y^*(T)$ is $y_o(T)$ ($\geqq 1$). Thus, only those values of T which are greater than T_ℓ will be considered for finding an optimal review period.

3.2 Order Level Optimized Net Revenue Function

The order level optimized net revenue function is defined to be (see (2.4))

$$(3.1) \qquad \begin{aligned} \pi(T) &\triangleq \pi(y_o(T),T) \\ &= rm(T) - k - G(y_o(T),T) \qquad \text{for all } T \in [T_0,\infty) \end{aligned}$$

and (see (2.5))

$$G(y_o(T),T) = cy_o(T) + (p + r) \sum_{n=y_o(T)+1}^{\infty} [n - y_o(T)]P_n(T)$$

$$(3.2) \qquad + \frac{h}{\alpha} \sum_{n=0}^{y_o(T)} [y_o(T) - n] \frac{I_{-q}(n + 1,\rho)}{\rho + n}$$

$$+ \frac{p}{\alpha} \sum_{n=y_o(T)+1}^{\infty} [n - y_o(T)] \frac{I_{-q}(n + 1,\rho)}{\rho + n} .$$

From (3.1) and (3.2), it is observed that the order level optimized net revenue is a function of the review period alone. Hence, an optimal review period can be sought by maximizing $\pi(T)$ with respect to T. Recall the fact (Proposition 2.8) that $y_o(T)$ is a step function of T, having jumps at points $T_0, T_1, \ldots, T_{\bar{N}-1}$ where at time T_N, $y_o(T)$ jumps from a value N to a vlue $N + 1$, being continuous from the right and $\bar{N}$ is the limiting value of $y_o(T)$.

Since the expressions for $\pi(T)$ involves $y_o(T)$, it will be interesting to see whether $\pi(T)$ has also jumps at $T_0, T_1, \ldots, T_{\bar{N}-1}$. It turns out that $\pi(T)$ is continuous for all values of T.

<u>Proposition 3.1</u>:

The order level optimized net revenue function is a continuous function of the review period.

<u>Proof</u>:

From Proposition 2.8, it follows that $y_o(T)$ is constant during the intervals $[T_0,T_1),[T_1,T_2), \ldots, \left[T_{\bar{N}-2},T_{\bar{N}-1}\right),\left[T_{\bar{N}-1},\infty\right)$. Denote by $\pi_N(T)$, the value of the function $\pi(T)$ during the interval $[T_{N-1},T_N)$ where $y_o(T)$ is equal to N. With the notation $T_{\bar{N}} = \infty$, N varies from $1,2, \ldots, \bar{N}$.

Hence,

$$\pi_N(T) = rm(T) - k - G(N,T) \qquad \text{for all } T_{N-1} \leq T < T_N \text{ and } N = 1,2, \ldots, \bar{N} \tag{3.3}$$

where

$$\begin{aligned} G(N,T) &= cN + (p + r) \sum_{n=N+1}^{\infty} (n - N)P_n(T) \\ &+ \frac{h}{\alpha} \sum_{n=0}^{N} (N - n) \frac{I_{\underline{q}}(n + 1,\rho)}{\rho + n} + \frac{\hat{p}}{\alpha} \sum_{n=N+1}^{\infty} (n - N) \frac{I_{\underline{q}}(n + 1,\rho)}{\rho + n} . \end{aligned} \tag{3.4}$$

It is clear from (3.3) and (3.4) that $\pi_N(T)$ is continuous in the interval $T_{N-1} \leq T < T_N$ for all $N = 1,2, \ldots, \bar{N}$. The only thing remains is to show that the jump at point T_N is zero. Denote

$$\Delta\pi_N(T_N) = \pi_{N+1}(T_N) - \pi_N(T_N) .$$

By (3.3), the jump at $T = T_N$, is

$$\begin{aligned} \Delta\pi_N(T_N) &= -G_{N+1}(T_N) + G_N(T_N) \\ &= -\Delta G_N(T_N) . \end{aligned}$$

In Section 2.13 (Corollary 3), it was shown that T_N satisfies $\Delta G_N(T_N) = 0$. Hence,

$$\Delta\pi_N(T_N) \equiv 0 \qquad \text{for all } N = 1,2, \ldots, (\bar{N}-1) .$$

3.3 Limiting Values of $\pi(T)$

The next question would be to examine the limiting values of $\pi(T)$ as T tends to zero and infinity.

(i) $T \to 0$:

When the review period tends to zero, $y_o(T) \to 0$ by (2.13). Now,

$$I_{\bar{q}}(n+1,\rho) = \frac{\int_0^{\bar{q}} u^n(1-u)^{\rho-1}du}{\int_0^1 u^n(1-u)^{\rho-1}du}$$

where

$$\bar{q} = 1 - e^{-\alpha T} .$$

As $T \to 0$, $\bar{q} \to 0$. Hence, $I_{\bar{q}}(n+1,\rho) \to 0$. Also, $m(T) \to 0$ and $P_n(T) \to 0$ for all $n \geq 1$. Hence, from (3.1),

$$\lim_{T\to 0} \pi(T) = 0 .$$

(ii) $T \to \infty$:

Now, as $T \to \infty$, $y_o(T) \to \bar{N}$ which is the finite upper bound for $y_o(T)$. Also, as $T \to \infty$, $\bar{q} \to 1$. Hence,

$$I_{\bar{q}}(n+1,\rho) \to 1 .$$

Also, $P_n(T) \to 0$ for all $n \geq 0$ and $m(T) \to \infty$. Now,

$$\lim_{T\to\infty} \pi(T) = \lim_{T\to\infty} [rm(T) - k - G(y_o(T),T)] .$$

From (3.2),

$$G(y_o(T),T) = cy_o(T) + (p + r) \sum_{n=y_o(T)+1}^{\infty} [n - y_o(T)]P_n(T)$$

$$+ \frac{h}{\alpha} \sum_{n=0}^{y_o(T)} [y_o(T) - n] \frac{I_{\bar{q}}(n + 1,\rho)}{\rho + n}$$

$$+ \frac{\hat{p}}{\alpha} \sum_{n=y_o(T)+1}^{\infty} [n - y_o(T)] \frac{I_{\bar{q}}(n + 1,\rho)}{\rho + n} .$$

Changing the order of summation of the terms containing $(p + r)$, we get,

$$G(y_o(T),T) = cy_o(T) + (p + r) \sum_{n=0}^{\infty} [n - y_o(T)]P_n(T)$$

$$- (p + r) \sum_{n=0}^{y_o(T)} [n - y_o(T)]P_n(T)$$

$$+ \frac{h}{\alpha} \sum_{n=0}^{y_o(T)} [y_o(T) - n] \frac{I_{\bar{q}}(n + 1,\rho)}{\rho + n}$$

$$+ \frac{\hat{p}}{\alpha} \sum_{n=y_o(T)+1}^{\infty} [n - y_o(T)] \frac{I_{\bar{q}}(n + 1,\rho)}{\rho + n} .$$

Using the fact that $\sum_{n=0}^{\infty} P_n(T) = 1$ and $\sum_{n=0}^{\infty} nP_n(T) = m(T)$, we get,

$$G(y_o(T),T) = cy_o(T) + (p + r)[m(T) - y_o(T)]$$

$$+ (p + r) \sum_{n=0}^{y_o(T)} [y_o(T) - n]P_n(T)$$

$$(3.5) \qquad + \frac{h}{\alpha} \sum_{n=0}^{y_o(T)} [y_o(T) - n] \frac{I_{\bar{q}}(n + 1,\rho)}{\rho + n}$$

$$+ \frac{\hat{p}}{\alpha} \sum_{n=y_o(T)+1}^{\infty} [n - y_o(T)] \frac{I_{\bar{q}}(n + 1,\rho)}{\rho + n} .$$

From (3.5), as $T \to \infty$,

$$cy_o(T) \to c\bar{N}$$

$$(p + r)[m(T) - y_o(T)] \to \infty$$

$$\frac{h}{\alpha}\sum_{n=0}^{y_o(T)} [y_o(T) - n] \frac{I_{\bar{q}}(n + 1,\rho)}{\rho + n} \to \frac{h}{\alpha}\sum_{n=0}^{\bar{N}} [\bar{N} - n] \frac{1}{\rho + n} < \infty$$

and positive

$$\frac{\hat{p}}{\alpha}\sum_{n=y_o(T)+1}^{\infty} [n - y_o(T)] \frac{I_{\bar{q}}(n + 1,\rho)}{\rho + n} \to \frac{\hat{p}}{\alpha}\sum_{n=\bar{N}+1}^{\infty} [n - \bar{N}] \frac{1}{\rho + n} = \infty .$$

Hence, $\lim_{T\to\infty} G(y_o(T),T)] = \infty$. Thus, $\lim_{T\to\infty} \pi(T) = \infty - \infty$ which is an indeterminate form. L'Hospital's Rule may be used to find the limit. Since the limiting value of $\pi(T)$ is not in the standard form ∞/∞ or $0/0$ for direct application of L'Hospital's Rule, transform the expression in such a way so as to get ∞/∞ form.

Multiply the expression (3.1) throughout by $\bar{p} = e^{-\alpha T}$ to obtain

$$\pi(T) = \frac{1}{\bar{p}} [r\rho\bar{q} - \bar{p}k - \bar{p}G(y_o(T),T)] .$$

This expression can be rewritten, using (3.5), as

$$\pi(T) = \frac{1}{\bar{p}}\Bigg\{-p\rho\bar{q} - k\bar{p}$$

$$+ \bar{p}\left[(p + r - c)y_o(T) - (p + r)\sum_{n=0}^{y_o(T)} [y_o(T) - n]P_n(T)\right]$$

$$- \bar{p}\left[\frac{h}{\alpha}\sum_{n=0}^{y_o(T)} [y_o(T) - n]\frac{I_{\bar{q}}(n + 1,\rho)}{\rho + n}\right]$$

$$- \left[\frac{\frac{\hat{p}}{\alpha} \sum_{n=y_o(T)+1}^{\infty} [n - y_o(T)] \frac{I_{\bar{q}}(n+1,\rho)}{\rho + n}}{1/\bar{p}} \right] \Bigg\}$$

$$= \frac{1}{\bar{p}} (Q_1 + Q_2 + Q_3 + Q_4)$$

where

$$Q_1 = -p\rho\bar{q} - k\bar{p}$$

$$Q_2 = \bar{p}\left[(p + r - c)y_o(T) - (p + r) \sum_{n=0}^{y_o(T)} [y_o(T) - n]P_n(T) \right]$$

$$Q_3 = -\bar{p}\left[\frac{h}{\alpha} \sum_{n=0}^{y_o(T)} [y_o(T) - n] \frac{I_{\bar{q}}(n+1,\rho)}{\rho + n} \right]$$

$$Q_4 = -\left[\frac{\frac{\hat{p}}{\alpha} \sum_{n=y_o(T)+1}^{\infty} [n - y_o(T)] \frac{I_{\bar{q}}(n+1,\rho)}{\rho + n}}{1/\bar{p}} \right].$$

As $T \to \infty$: $y_o(T) \to \bar{N}$; $\dfrac{I_{\bar{q}}(n+1,\rho)}{\rho + n} \to \dfrac{1}{\rho + n}$; $Q_1 \to -p\rho$; $Q_2 \to 0$; $Q_3 \to 0$ and $Q_4 \to \infty/\infty$. So apply L'Hospital's Rule for Q_4 . Hence,

$$\lim_{T\to\infty} [Q_4] = \lim_{T\to\infty} \left[\frac{-\hat{p} \sum_{n=y_o(T)+1}^{\infty} [n - y_o(T)]P_n(T)}{\alpha/\bar{p}} \right].$$

Changing the order of summation,

$$\lim_{T\to\infty} [Q_4] = \lim_{T\to\infty} \left[\frac{-\hat{p}m(T) + \hat{p}y_o(T) - \hat{p} \sum_{n=0}^{y_o(T)} [y_o(T) - n]P_n(T)}{\alpha/\bar{p}} \right]$$

$$= \lim_{T\to\infty}\left\{-\frac{\hat{p}\rho\bar{q}}{\alpha} - \frac{\bar{p}}{\alpha}\left[\hat{p}\sum_{n=0}^{y_o(T)}[y_o(T) - n]P_n(T) - \hat{p}y_o(T)\right]\right\}$$

$$= -\hat{p}\rho/\alpha \ .$$

Hence, the numerator of $\pi(T)$, which is nothing but $(Q_1 + Q_2 + Q_3 + Q_4)$ tends to a constant, $-(p + \hat{p}/\alpha)\rho$, while the denominator $\bar{p}$ tends to zero. Hence, $\lim_{T\to\infty} \pi(T) = -\infty$.

3.4 Derivative of $\pi(T)$

In expressions (3.1) and (3.2), $\pi(T)$ involves the step function $y_o(T)$ which does not have left-hand derivatives at points $T_0, T_1, \ldots, T_{\bar{N}-1}$. Hence, fixing the value of $y_o(T) = N$ where it is constant for all $T_{N-1} \leq T < T_N$, its derivative may be examined with respect to $T \in [T_{N-1}, T_N)$. From (3.3) and (3.4),

$$\pi_N(T) = rm(T) - k$$

$$- cN - (p + r)\sum_{n=N+1}^{\infty}(n - N)P_n(T)$$

$$- \frac{h}{\alpha}\sum_{n=0}^{N}(N - n)\frac{I_{\bar{q}}(n + 1,\rho)}{\rho + n} - \frac{\hat{p}}{\alpha}\sum_{n=N+1}^{\infty}(n - N)\frac{I_{\bar{q}}(n + 1,\rho)}{\rho + n}$$

for all $T_{N-1} \leq T < T_N$ and for all $N = 1,2, \ldots, \bar{N}$.

(Note: $T_{\bar{N}} = +\infty$.)

$$\pi_N'(T) = \frac{d}{dT}[\pi_N(T)] \ .$$

Using (2.1) and (2.19),

$$\pi_N'(T) = \frac{\lambda r}{\bar{p}} - (p + r) \sum_{n=N+1}^{\infty} (n - N)[(\rho + n - 1)\bar{p}\alpha P_{n-1}(T) - \lambda P_n(T)]$$

$$- h \sum_{n=0}^{N} (N - n)P_n(T) - \hat{p} \sum_{n=N+1}^{\infty} (n - N)P_n(T)$$

change the order of summation for $\hat{p}$ term to obtain,

$$\pi_N'(T) = \frac{\lambda r}{\bar{p}} - (h + \hat{p}) \sum_{n=0}^{N} (N - n)P_n(T) + \hat{p}[N - m(T)]$$

(3.6) $$- (p + r) \sum_{n=N+1}^{\infty} (n - N)[(\rho + n - 1)\bar{p}\alpha P_{n-1}(T) - \lambda P_n(T)]$$

for all $T_{N-1} \leq T < T_N$ and for all $N = 1,2, \ldots, \bar{N}$.

Proposition 3.2:

For all $N = 1,2, \ldots, \bar{N}$, the function $\pi_N'(T)$ is continuous in $T \in [T_{N-1},T_N)$ and has a positive jump at the boundary T_N where $y_o(T)$ changes its value from N to $(N + 1)$.

Proof:

From (3.6), it is obvious that $\pi_N'(T)$ is continuous in the interior. The only thing is to prove the jump at $T = T_N$ is positive.

$$\Delta\pi_N'(T_N) = \text{jump of the derivative } \pi_N'(T) \text{ at } T = T_N < \infty$$

$$= \pi_{N+1}'(T_N) - \pi_N'(T_N) .$$

From (3.6), the above will be equal to

$$\Delta\pi_N'(T_N) = \hat{p} - (h + \hat{p})\left[\sum_{n=0}^{N+1} (N + 1 - n)P_n(T_N) - \sum_{n=0}^{N} (N - n)P_n(T_N)\right]$$

$$- (p + r)\left[\sum_{n=N+2}^{\infty} (n - N - 1)][(\rho + n - 1)\bar{p}\alpha P_{n-1}(T_N) - \lambda P_N(T_N)]\right.$$

$$\left.- \sum_{n=N+1}^{\infty} (n - N)[(\rho + n - 1)\bar{p}\alpha P_{n-1}(T_N) - \lambda P_n(T_N)]\right] .$$

Expanding the terms under summation and after cancellation,

$$\Delta\pi_N'(T_N) = \hat{p} - (h + \hat{p}) \sum_{n=0}^{N} P_n(T_N)$$

$$- (p + r) \sum_{n=0}^{N} [(\rho + n - 1)\bar{p}\alpha P_{n-1}(T_N) - \lambda P_n(T_N)] .$$

The above expression and (2.20) are identical except for a sign change. Hence, $\Delta\pi_N'(T_N) = -F_N'(T_N)$. From Corollaries 2 and 3, in Chapter 2,

(3.7) $$F_N'(T) < 0 \qquad \text{for all } T \geq T_N$$

since $F_N(T)$ crosses the "$(p + r - c)$ line" at T_N and continues to (strictly) decrease. Hence, $\Delta\pi_N'(T_N) > 0$.

It will be easy to investigate the behavior of $\pi_N'(T)$ if there are finite limits in the summation. Hence, by changing the infinite limits in the terms corresponding to "$(p + r)$" and "$\hat{p}$" in expression (3.6) to finite limits, (3.6) can be rewritten as,

$$\pi_N'(T) = \frac{\lambda r}{\bar{p}} - (h + \hat{p}) \sum_{n=0}^{N} [N - n]P_n(T) + \hat{p}[N - m(T)]$$

$$+ \lambda(p + r)\left[\left(\frac{\rho\bar{q}}{\bar{p}} - \sum_{n=0}^{N} nP_n(T)\right) - N\left(1 - \sum_{n=0}^{N} P_n(T)\right)\right]$$

$$- \alpha(p + r)\bar{p}\left[\sum_{n=N+1}^{\infty} (n - N)[\rho P_{n-1}(T) + (n - 1)P_{n-1}(T)]\right] .$$

Let

$$L(T) = -\alpha(p + r)\bar{p}\left[\sum_{n=N+1}^{\infty} (n - N)[\rho P_{n-1}(T) + (n - 1)P_{n-1}(T)]\right]$$

$$= \alpha(p + r)\bar{p}\left[N\left(\rho - \rho \sum_{n=1}^{N} P_{n-1}(T)\right) + N\left(m(T) - \sum_{n=1}^{N} (n - 1)P_{n-1}(T)\right)\right]$$

$$- \alpha(p + r)\bar{p}\left[\sum_{n=N+1}^{\infty} \left\{\rho[(n - 1)P_{n-1}(T) + P_{n-1}(T)]\right.\right.$$

$$\left.\left.+ [(n - 1)^2 + (n - 1)]P_{n-1}(T)\right\}\right] .$$

Once again changing the infinite limits in the summation to finite limits and using the fact that

$$\sum_{n=0}^{\infty} n^2 P_n(T) = V(T) + [m(T)]^2 = \frac{\rho^2\bar{q}^2 + \rho\bar{q}}{\bar{p}^2} ,$$

we have

$$L(T) = \alpha(p + r)\bar{p}\left\{N[\rho + m(T)] - N \sum_{n=1}^{N} (\rho + n - 1)P_{n-1}(T)\right.$$

$$-\left[\rho\, m(T) + \rho/\bar{p} - \rho \sum_{n=1}^{N} nP_{n-1}(T) - \sum_{n=1}^{N} n(n - 1)P_{n-1}(T)\right.$$

$$\left.\left.+ [m(T)]^2 + \frac{m(T)}{\bar{p}}\right]\right\} ,$$

i.e.,

$$L(T) = -\alpha(p + r)\bar{p}\left\{-N[\rho + m(T)] + \sum_{n=1}^{N} [N - n](\rho + n - 1)P_{n-1}(T)\right.$$

$$\left.+ \frac{\rho\bar{q}}{\bar{p}}\left[\rho + \frac{1}{\bar{q}} + \frac{\rho\bar{q}}{\bar{p}} + \frac{1}{\bar{p}}\right]\right\} .$$

Hence,

$$\pi_N'(T) = \frac{\lambda r}{\bar{p}} - (h + \hat{p}) \sum_{n=0}^{N} [N - n]P_n(T) + \hat{p}[N - m(T)]$$

$$+ \lambda(p + r)\left[m(T) - N + \sum_{n=0}^{N} (N - n)P_n(T)\right]$$

$$- \alpha(p + r)\bar{p}\left[\sum_{n=1}^{N} (N - n)(\rho + n - 1)P_{n-1}(T)\right]$$

$$+ \alpha(p + r)\left[\bar{p}N\rho + \rho N\bar{q} - \frac{\rho\bar{q}}{\bar{p}}\left(\rho + \frac{1}{\bar{q}}\right)\right].$$

The above reduces finally to

$$\pi_N'(T) = \frac{-\lambda p}{\bar{p}} + \hat{p}[N - m(T)] - (h + \hat{p}) \sum_{n=0}^{N} [N - n]P_n(T)$$

$$(3.8) \qquad - (p + r) \sum_{n=0}^{N} [N - n][(\rho + n - 1)\bar{p}\alpha P_{n-1}(T) - \lambda P_n(T)]$$

for all $T_{N-1} \leqq T < T_N$ and for all $N = 1,2, \ldots, \bar{N}$.

Now, a general expression for $\pi'(T)$ may be written by replacing N by $y_0(T)$.

$$\pi'(T) = \frac{-\lambda p}{\bar{p}} + \hat{p}[y_0(T) - m(T)] - (h + \hat{p}) \sum_{n=0}^{y_0(T)} [y_0(T) - n]P_n(T)$$

$$(3.9) \qquad - (p + r) \sum_{n=0}^{y_0(T)} [y_0(T) - n][(\rho + n - 1)\bar{p}\alpha P_{n-1}(T) - \lambda P_n(T)]$$

for all $T_{y_0(T)-1} \leqq T < T_{y_0(T)}$.

3.5 Properties of $\pi(T)$

To find the best review period which maximizes $\pi(T)$, it is necessary to know the behavior of $\pi(T)$ as T increases in the range $[T_0,\infty)$. This, in turn, leads to the investigation of $\pi'(T)$. Since, the derivative is defined only in the intervals where $y_o(T)$ is constant, each of the intervals is investigated separately. Note that there are only a finite number of intervals. As a matter of fact, there are only $\bar{N}$ intervals since $y_o(T)$ is bounded above by $\bar{N}$.

For all fixed $y_o(T) = N$ in the range $T_{N-1} \leq T < T_N$, expression (3.8) may be written as,

$$\pi_N'(T) = \frac{-\lambda p - \rho\hat{p} + S_N(T)}{\bar{p}} \tag{3.10}$$

where

$$\begin{aligned} S_N(T) = \bar{p}\Bigg[(N + \rho)\hat{p} - (h + \hat{p}) \sum_{n=0}^{N} (N - n)P_n(T) \\ - (p + r) \sum_{n=0}^{N} [N - n][(\rho + n - 1)\bar{p}\alpha P_{n-1}(T) - \lambda P_n(T)]\Bigg] \end{aligned} \tag{3.11}$$

$$\text{for all } T_{N-1} \leq T < T_N$$

(3.10) and (3.11) hold for all $N = 1,2, \ldots, \bar{N}$. Now investigate $S_N(T)$ with respect to $T \in [T_{N-1}, T_N)$. Once again, make use of the transformation $u = \bar{q} = 1 - e^{-\alpha T}$ and $P_n(T) = c_n(1 - u)^\rho u^n$ where $c_n = \frac{\Gamma(\rho + n)}{\Gamma(\rho)\Gamma(n + 1)}$. Since $u_N = 1 - e^{-\alpha T_N}$, the problem reduces to examining $S_N(u)$ with respect to $u \in [u_{N-1}, u_N)$. Thus, from (3.10),

$$\begin{aligned} S_N(u) = (1 - u)\Bigg[\hat{p}(N + \rho) \\ + (1 - u)^\rho\Bigg\{-\alpha(p + r) \sum_{n=1}^{N} (N - n)(\rho + n - 1)c_{n-1}u^{n-1}(1 - u) \\ + [\lambda(p + r) - (h + \hat{p})] \sum_{n=0}^{N} (N - n)c_n u^n\Bigg\}\Bigg]. \end{aligned}$$

Consider only the terms within the curly brackets which may be rewritten using the fact $(\rho + n - 1)c_{n-1} \equiv nc_n$ as,

$$\{\ \} = -\alpha(p + r) \sum_{n=1}^{N} (N - n)(\rho + n - 1)c_{n-1}u^{n-1}$$

$$+ \sum_{n=0}^{N} [N - n]u^n[\alpha(p + r)n + \lambda(p + r) - (h + \hat{p})]c_n \ .$$

Collecting the coefficients of $u^0, u^1, \ldots, u^N$ separately and simplifying,

$$\{\ \} = \alpha(p + r) \sum_{n=0}^{N-1} nc_n u^n + \lambda(p + r) \sum_{n=0}^{N-1} c_n u^n$$

$$- (h + \hat{p}) \sum_{n=0}^{N-1} (N - n)c_n u^n \ .$$

Hence,

$$S_N(u) = (1 - u)\left[\hat{p}(N + \rho) + (1 - u)^\rho\left[(p + r) \sum_{n=0}^{N-1} (\alpha n + \lambda)c_n u^n\right.\right.$$

$$\left.\left. - (h + \hat{p}) \sum_{n=0}^{N-1} (N - n)c_n u^n\right]\right] \tag{3.12}$$

for all $u_{N-1} \leq u < u_N$ and for all $N = 1,2, \ldots, \bar{N}$.

Fact 1:

As $u \to 1$, $S_N(u) \to 0$. Hence, $S_N(T) \to 0$ as $T \to \infty$ for all values of $N = 1,2, \ldots, \bar{N}$.

Fact 2:

From (3.7),

$$F'_N(T) < 0 \qquad \text{for all } T \geq T_N$$

$$\iff$$

$$F'_N(u) < 0 \qquad \text{for all } u \geq u_N \ .$$

Thus, from (2.24), it follows that

$$F'_N(u) = -\hat{p} + (1 - u)^{\rho}\left[(h + \hat{p}) \sum_{n=0}^{N} c_n u^n - (\lambda + \alpha N)(p + r)c_N u^N\right] < 0 \tag{3.13}$$

for all $u \geq u_N$ and for all $N = 0,1,2, \ldots, \bar{N}$.

<u>Proposition 3.3</u>:

For all $N = 1,2, \ldots, \bar{N}$, the function $S_N(T)$ (as defined in (3.11)) (strictly) decreases as T increases in the range $[T_{N-1}, T_N)$.

<u>Proof</u>:

To show $S_N(T)$ is strictly decreasing in $T \in [T_{N-1}, T_N)$, it is sufficient to show that $S'_N(T) < 0$ for all $T \in [T_{N-1}, T_N)$, equivalently, $S'_N(u) < 0$ for all $u \in [u_{N-1}, u_N)$ since $S'_N(T) = S'_N(u)\alpha(1 - u)$. Now,

$$\begin{aligned} S'_N(u) = \frac{dS_N(u)}{du} &= -(N + \rho)\hat{p} \\ &+ (1 - u)^{\rho+1}\left[\alpha(p + r) \sum_{n=1}^{N-1} nc_n nu^{n-1} + \lambda(p + r) \sum_{n=1}^{N-1} nc_n u^{n-1}\right. \\ &\left. - (h + \hat{p}) \sum_{n=1}^{N-1} (N - n)nc_n u^{n-1}\right] \\ &- (\rho + 1)(1 - u)^{\rho}\left[\alpha(p + r) \sum_{n=1}^{N-1} nc_n u^n + \lambda(p + r) \sum_{n=0}^{N-1} c_n u^n\right. \\ &\left. - (h + \hat{p}) \sum_{n=0}^{N-1} (N - n)c_n u^n\right] . \end{aligned}$$

Taking $(1 - u)^{\rho}$ as a common factor and using the fact $nc_n = (\rho + n - 1)c_{n-1}$, we get after rearranging the terms,

$$S_N'(u) = -(N+\rho)\hat{p} + (1-u)^{\rho}\Bigg\{\Bigg[\alpha(p+r)\sum_{n=1}^{N-1}(\rho+n-1)c_{n-1}nu^{n-1}$$

$$-\alpha(p+r)\sum_{n=1}^{N-1}nc_n nu^n\Bigg]$$

$$+\Bigg[\lambda(p+r)\sum_{n=1}^{N-1}(\rho+n-1)c_{n-1}u^{n-1} - \lambda(p+r)\sum_{n=1}^{N-1}nc_n u^n\Bigg]$$

$$-\Bigg[(h+\hat{p})\sum_{n=1}^{N-1}(N-n)(\rho+n-1)c_{n-1}u^{n-1} - (h+\hat{p})\sum_{n=1}^{N-1}(N-n)nc_n u^n\Bigg]$$

$$-\Bigg[(\rho+1)\alpha(p+r)\sum_{n=1}^{N-1}nc_n u^n + (\rho+1)\lambda(p+r)\sum_{n=0}^{N-1}c_n u^n\Bigg]$$

$$+\Bigg[(\rho+1)(h+\hat{p})\sum_{n=0}^{N-1}(N-n)c_n u^n\Bigg]\Bigg\}$$

concentrating only on the terms within the curly brackets, collect the coefficients of $u^0, u^1, \ldots, u^{N-1}$ and after simplifying,†

$$S_N'(u) = (N+\rho)\Bigg\{(1-u)^{\rho}\Bigg[(h+\hat{p})\sum_{n=0}^{N-1}c_n u^n - [\lambda+\alpha(N-1)](p+r)c_{N-1}\Bigg] - \hat{p}\Bigg\} \tag{3.14}$$

for all $u_{N-1} \leq u < u_N$ and for all $N = 1,2,\ldots,\bar{N}$.

Since (3.13) is true for all $N = 0,1,2,\ldots,\bar{N}$, it is true in particular for $N = N - 1 \in [0,\bar{N}]$ also. Thus, from (3.13) at $N = (N-1)$,

†This process is little bit involved. We suggest writing the coefficients u^n separately for all $n = 0,1,2,\ldots, N-1$ and then cancel the terms using the fact $\rho = \lambda/\alpha$.

$$F'_{N-1}(u) = -\hat{p} + (1-u)^{\rho}\Bigg[(h+\hat{p})\sum_{n=0}^{N-1} c_n u^n$$

$$- [\lambda + \alpha(N-1)](p+r)c_{N-1}u^{N-1}\Bigg] < 0 \quad \text{for all} \quad u \geqq u_{N-1} .$$

Thus,

$$S'_N(u) < 0 \qquad \text{for all} \quad u \geqq u_{N-1} . \tag{3.15}$$

In particular,

$$S'_N(u) < 0 \qquad \text{for} \quad u \in [u_{N-1}, u_N] .$$

Since $\alpha > 0$ and $(1-u) > 0$ for all $T \geqq T_o$ (> 0) ,

$$S'_N(T) < 0 \qquad \text{for all} \quad T \in [T_{N-1}, T_N)$$

$$\text{and for all} \quad N = 1,2, \ldots, \bar{N} .$$

<u>Proposition 3.4</u>:

For all $N = 1,2, \ldots, (\bar{N}-1)$, the function $S_N(T)$ has a positive jump at $T = T_N < \infty$.

<u>Proof</u>:

It is sufficient to show $S_N(u)$ has a positive jump at $u = u_N < 1$

$$\Delta S_N(u_N) = S_{N+1}(u_N) - S_N(u_N) .$$

By (3.12),

$$= \hat{p}(1 - u_N) + (1 - u_N)^{\rho+1}\left\{\alpha(p + r)\left[\sum_{n=0}^{N} nc_n u_N^n - \sum_{n=0}^{N-1} nc_n u_N^n\right]\right.$$

$$+ \lambda(p + r)\left[\sum_{n=0}^{N} c_n u_N^n - \sum_{n=0}^{N-1} c_n u_N^n\right]$$

$$\left. - (h + \hat{p})\left[\sum_{n=0}^{N} (N + 1 - n)c_n u_N^n - \sum_{n=0}^{N-1} (N - n)c_n u_N^n\right]\right\} .$$

Cancelling some of the terms after expansion,

$$\Delta S_N(u_N) = (1 - u_N)\left\{(1 - u_N)^{\rho}\left[(\lambda + \alpha N)(p + r)c_N u_N^N\right.\right.$$

$$\left.\left. - (h + \hat{p}) \sum_{n=0}^{N} c_n u_N^n\right] + \hat{p}\right\} .$$

By (3.13),

$$(1 - u)^{\rho}\left[(h + \hat{p}) \sum_{n=0}^{N} c_n u^n - (\lambda + \alpha N)(p + r)c_N u^N\right] - \hat{p} < 0$$

for $u \geq u_N$ and for all $N = 0,1,2, \ldots, \bar{N}$.

Thus,

$$\Delta S_N(u_N) > 0 \qquad \text{for all } N = 1,2, \ldots, (\bar{N}-1) \text{ and } u_N < 1 .$$

<u>Proposition 3.5</u>:

For $N = 1,2, \ldots, \bar{N}$ (finite),

(i) $\pi_N'(T)$ can have at most one change in sign in the range $T_{N-1} \leq T < T_N$.

(ii) In particular, if $\pi_N'(T) \leq 0$ for some $\hat{T} \in [T_{N-1}, T_N)$, then $\pi_N(T)$ is strictly concave thereafter up to T_N .

Proof:

(i) From (3.10),

$$\pi_N'(T) = \frac{-\lambda p - \rho\hat{p} + S_N(T)}{\bar{p}} .$$

By Proposition 3.3, $S_N(T)$ strictly decreases in the range $T_{N-1} \leqq T < T_N$. Hence, once the numerator becomes negative in that interval, it stays negative in that interval, since $\bar{p} > 0$ and $(-\lambda p - \rho\hat{p})$ is a constant.

(ii) Assume $\pi_N'(\hat{T}) \leqq 0$ for $\hat{T} \in [T_{N-1}, T_N)$

$$\pi_N''(T) = \pi_N''(u)(1 - u)\alpha ,$$

since $\pi_N'(T) \equiv \pi_N'(u)$. From (3.10),

$$\pi_N'(u) = \frac{-\lambda p - \rho\hat{p} + S_N(u)}{1 - u} .$$

Hence,

$$\pi_N''(T) = \frac{[(1 - u)S_N'(u) + (-\lambda p - \rho\hat{p} + S_N(u))](1 - u)\alpha}{(1 - u)^2}$$

for all $T_{N-1} \leqq T < T_N$.

By (3.15), $S_N'(u) < 0$ for all $u \geqq u_{N-1}$. By assumption $\pi_N'(\hat{T}) \leqq 0$

$$\Rightarrow -\lambda p - \rho\hat{p} + S_N(\hat{u}) \leqq 0 .$$

Hence,

$$\pi_N''(T) < 0 \qquad \text{for all } \hat{T} \leq T < T_N$$

$\Rightarrow \pi_N(T)$ is strictly concave in the range $\hat{T} \leq T < T_N$.

Corollary 1:

For $N = 1,2, \ldots, \bar{N}$, $\pi_N(T)$ attains its maximum value for a unique $T \in [T_{N-1}, T_N)$.

Proof:

Follows directly from Proposition (3.5).

Proposition 3.6:

The expression $S_N(T)$ (refer to (3.11)) is strictly positive for all $N = 1,2, \ldots, \bar{N}$ and $T \in [T_{N-1}, T_N)$.

Proof:

From (3.11) and (3.12),

$$S_N(T) = S_N(u)$$

$$= (1 - u)\left[\hat{p}(N + \rho) + (1 - u)^\rho \left[(p + r) \sum_{n=0}^{N-1} (\alpha n + \lambda) c_n u^n \right.\right.$$

$$\left.\left. - (h + \hat{p}) \sum_{n=0}^{N-1} (N - n) c_n u^n \right]\right]$$

for all $u_{N-1} \leq u < u_N$ and $N = 1,2, \ldots, \bar{N}$

By expanding the terms, the following identity may be verified

$$\sum_{n=0}^{N-1} (N - n) c_n u^n \equiv \sum_{n=0}^{N-1} c_n u^n + \sum_{n=0}^{N-2} c_n u^n + \ldots$$

$$+ \sum_{n=0}^{2} c_n u^n + \sum_{n=0}^{1} c_n u^n + c_o u^o .$$

Thus, $S_N(u)$ may be rewritten using the above identity as,

$$S_N(u) = (1 - u)\left[\hat{p}(N + \rho) + (1 - u)^{\rho}\left\{(p + r)\sum_{n=0}^{N-1} (\alpha n + \lambda)c_n u^n - (h + \hat{p})\sum_{m=0}^{N-1}\left(\sum_{n=0}^{m} c_n u^n\right)\right\}\right]$$

for $u_{N-1} \leq u < u_N$ and $N = 1,2, \ldots, \bar{N}$.

After rearranging the terms,

$$S_N(u) = (1 - u)\left[\rho\hat{p} + \sum_{m=0}^{N-1} \hat{p} + (1 - u)^{\rho}\left[(p + r)(\alpha m + \lambda)c_m u^m - (h + \hat{p})\sum_{n=0}^{m} c_n u^n\right]\right] \tag{3.16}$$

for all $u_{N-1} \leq u < u_N$ and $N = 1,2, \ldots, \bar{N}$.

Since $u_n = 1 - e^{-\alpha T_n}$ for all $n = 0,1,2, \ldots, \bar{N}-1$, and $T_0 < T_1 < \ldots < T_{\bar{N}-1}$ (refer to Corollary 3), $u_0 < u_1 < \ldots < u_{\bar{N}-1}$. Hence, using (3.13), for all $u_{N-1} \leq u < u_N$,

$$\sum_{m=0}^{N-1} \hat{p} + (1 - u)^{\rho}\left[(p + r)(\alpha m + \lambda)c_m u^m - (h + \hat{p})\sum_{n=0}^{m} c_n u^n\right] > 0 \ . \tag{3.17}$$

Using (3.17) and the fact $\rho\hat{p} \geq 0$ and $(1 - u) > 0$,

$$S_N(u) > 0 \qquad \text{for all } u_{N-1} \leq u < u_N \text{ and } N = 1,2, \ldots, \bar{N}$$

=>

$$S_N(T) > 0 \qquad \text{for all } T_{N-1} \leq T < T_N \text{ and } N = 1,2, \ldots, \bar{N} \ . \tag{3.18}$$

Proposition 3.7:

There exists a finite lower bound $T_\ell > T_o$ for the review period such that

$$k + G(y_o(T),T) \gtreqless G(0,T) \qquad \text{for } T \lesseqgtr T_\ell$$

for finite set-up cost k .

Proof:

By (2.13), $y_o(T) = 0$ for $0 \leq T < T_o$

$$\Rightarrow k + G(y_o(T),T) \geq G(0,T) \qquad \text{for } 0 \leq T < T_o ,$$

since $G(y_o(T),T) \equiv G(0,T)$. Hence, $T_\ell \notin [0,T_o)$. So, consider only positive order levels, i.e., $y_o(T) \geq 1$ ($\Rightarrow T \geq T_o$) . At $T = T_o$, $G(y_o(T),T) = G(1,T_o) = G(0,T_o)$ since $\Delta G(0,T_o)$ is zero. (Refer to Section 2.13, Corollary 3.)

Let $G'(y_o(T),T) = \dfrac{dG(y_o(T),T)}{dT}$. From (3.1),

$$G'(y_o(T),T) = \frac{r\lambda}{\bar{p}} - \pi'(T) \qquad \text{for all } T_o \leq T < \infty . \tag{3.19}$$

Note:

In (3.19), only right-hand derivatives exist at $T_0, T_1, \ldots, T_{\bar{N}-1}$ since $y_o(T)$ has finite saltus at these points and is right continuous.

Substituting the value of $\pi'(T)$ from (3.10) and using the fact $y_o(T) = N$ for $T_{N-1} \leq T < T_N$,

$$G'(y_o(T),T) = G'(N,T) = \frac{\lambda(p + r + \hat{p}/\alpha) - S_N(T)}{\bar{p}} \tag{3.20}$$

$$\text{for } T_{N-1} \leq T < T_N \text{ and } N = 1,2, \ldots, \bar{N} .$$

Note:

$T_{\bar{N}} = +\infty$.

From (2.10),

$$G(0,T) = (p + r + \hat{p}/\alpha)m(T) - \hat{p}\rho T$$

(3.21) $$\Rightarrow G'(0,T) = \frac{\lambda(p + r + \hat{p}/\alpha) - \hat{p}\rho\bar{p}}{\bar{p}} > 0 .$$

Hence, $G(0,T)$ strictly increases in T . From (3.20) and (3.21),

$$G'(0,T) - G'(N,T) = \frac{-\hat{p}\rho\bar{p} + S_N(T)}{\bar{p}} \qquad \text{for all } T_{N-1} \leqq T < T_N$$

and $N = 1,2, \ldots, \bar{N}$.

Using the transformation $u = 1 - e^{-\alpha T}$,

$$G'(0,u) - G'(N,u) = \frac{-\hat{p}\rho(1 - u) + S_N(u)}{(1 - u)} \qquad \text{for all } u_{N-1} \leqq u < u_N$$

and $N = 1,2, \ldots, \bar{N}$.

By (3.16) and (3.17),

$$S_N(u) - \hat{p}\rho(1 - u) > 0 \qquad \text{for all } u_{N-1} \leqq u < u_N$$

and $N = 1,2, \ldots, \bar{N}$

$\Rightarrow$

$$G'(0,u) - G'(N,u) > 0 \qquad \text{for all } u_{N-1} \leqq u < u_N$$

and $N = 1,2, \ldots, \bar{N}$.

Hence,

$$(3.22) \qquad G'(0,T) - G'(y_o(T),T) > 0 \qquad \text{for all} \quad T \in [T_o,\infty)$$

since $y_o(T) = N$ for $T_{N-1} \leq T < T_N$ and $N = 1,2, \ldots, \bar{N}$. Since $G(y_o(T),T)$ and $G(0,T)$ have the same value at $T = T_o$ and the rate of increase of $G(0,T)$ is strictly greater than that of $G(y_o(T),T)$ (by (3.22)), the difference $[G(0,T) - G(y_o(T),T)]$ strictly increases in T . Once the difference equals the finite set-up cost k at $T = T_\ell$, it will continue to increase (strictly) for all $T > T_\ell$ $(> T_o)$, thus proving the proposition.

Comment 1:

A positive inventory level is required to remain in business, i.e., $y_o(T) \geq 1$. Thus, the value of the lower bound on the review period T_ℓ is necessary since $y^*(T) \equiv 0$ for $T \in [0,T_\ell]$. Let N_ℓ denote $y_o(T_\ell)$. Hence, an optimal review period that exceeds T_ℓ is sought so as to get an order level at least equal to $N_\ell (\geq 1)$.

Note:

$$(3.23) \qquad y_o(T) = N_\ell \qquad \text{for} \quad T_\ell \leq T < T_{N_\ell} .$$

3.6 Algorithm to Compute Optimal Review Period

It was shown in Section 3.2 that $\pi(T)$ tends to $-\infty$ as $T \to \infty$. So, $\pi(T)$ reaches its maximum value at some $T < \infty$. Corollary 1 to Proposition 3.5 implies, for $N = 1,2, \ldots, \bar{N}$, there exists a unique T for every interval $[T_{N-1},T_N)$ where $\pi_N(T)$ attains its maximum in this interval. Since the number of intervals are finite, the problem reduces to just finding the maximum value of $\pi_N(T)$ for each $N = N_\ell, N_{\ell+1}, \ldots, \bar{N}$ and choosing the best of all the maximums. (Refer to Comment 1 and (3.23) at the end of Section 3.5.)

Let $\hat{T}_N$ denote the point in the interval $[T_{N-1}, T_N)$ where $\pi_N(T)$ attains its maximum value. Any one of the following three situations may arise:

(i) $\pi_N'(T_{N-1}) \leq 0$.

By Proposition 3.5, $\pi_N(T)$ decreases in this interval. Hence, $\hat{T}_N$ is T_{N-1} .

(ii) $\pi_N'(T_{N-1}) > 0$ and $\pi_N'(T_N) > 0$.

Again by Proposition (3.5), $\pi_N'(T) > 0$ for all $T \in [T_{N-1}, T_N)$. Hence, $\pi_N(T)$ strictly increases in this interval and $\hat{T}_N = T_N$.

(iii) $\pi_N'(T_{N-1}) > 0$ and $\pi_N'(T_N) < 0$.

By Proposition (3.5) and Corollary 1, there exists a unique $\hat{T}_N$ where $\pi_N'(\hat{T}_N) = 0$. Hence, the maximum value is attained at $\hat{T}_N$.

A step by step algorithm to compute the optimal review period is given which can be programmed directly on a digital computer.

<u>Algorithm</u>:

Denote the optimal review period by $\bar{T}$ and the optimum value of the order level optimized net return function by $\bar{\pi}$. Let $N = N_\ell$.

<u>Initializing Step</u>: (Refer to Comment 1 and (3.23) at the end of Section 3.5.)

Compute $\pi'(T_\ell)$. If $\pi'(T_\ell) \leq 0$, then set $\bar{\pi} = \pi(T_\ell)$, $\bar{T} = T_\ell$ and go to Step 2.

If $\pi'(T_\ell) > 0$, there are two possibilities:

(i) $\pi'(T_\ell) > 0$, $\pi'\left(T_{N_\ell}\right) > 0$. Then set $\bar{\pi} = \pi\left(T_{N_\ell}\right)$, $\bar{T} = T_{N_\ell}$ and go to Step 2.

(ii) $\pi'(T_\ell) > 0$, $\pi'\left(T_{N_\ell}\right) \leq 0$. Then, seek a $\hat{T} \in \left[T_\ell, T_{N_\ell}\right)$ such that $\pi'(T)$ is zero. Set $\bar{\pi} = \pi(\hat{T})$, $\bar{T} = \hat{T}$ and go to Step 2.

Step 1:

Compute $\pi_N'(T_{N-1})$. If $\pi_N'(T_{N-1}) \leq 0$, go to Step 2. Otherwise, go to Step 3.

Step 2:

Increase N by one. If $N \leq \bar{N}$, go to Step 1. Otherwise, go to Step 5.

Step 3:

Compute $\pi_N'(T_N)$. If $\pi_N'(T_N) < 0$, go to Step 4. Otherwise, set $\bar{\pi} = \pi_N(T_N)$, $\bar{T} = T_N$ and go to Step 2.

Step 4:

Compute $\hat{T}$ in the range $T_{N-1} \leq \hat{T} < T_N$ such that $\pi_N'(\hat{T}) = 0$. (Since in this case $\pi_N'(T_{N-1}) > 0$ and $\pi_N'(T_N) < 0$, the bisection procedure described in Chapter 2 may be used to find $\hat{T}$.) Compute $\pi_N(\hat{T})$. If $\pi_N(\hat{T}) \leq \bar{\pi}$, go to Step 2. Otherwise, set $\bar{\pi} = \pi_N(\hat{T})$, $\bar{T} = \hat{T}$ and go to Step 2.

Step 5:

Terminate. The current value of $\bar{T}$ is the optimal review period and $\bar{\pi}$ is the maximum value of the profit function.

Finiteness of Algorithm

By Proposition 2.6, the order level is bounded as $T \to \infty$ (i.e., $\bar{N}$ is finite). Hence, only a finite number of intervals $[T_{N-1}, T_N)$ for N

ranging from $N_\ell, N_{\ell+1}, \ldots, \bar{N}$ are to be examined for the maximum value of $\pi(T)$. Hence, the algorithm terminates in a finite number of steps.

3.7 Case When "Lost Sales" Costs are Ignored

In Section 3.0, it was mentioned that the cost of "lost sales" may be ignored by making p and $\hat{p}$ zero. This may simplify the cost and revenue expressions considerably. Since p and $\hat{p}$ are assumed to be nonnegative throughout the analysis so far, all the results of optimal order policy and optimal review period hold for this case also. An example given in Figure 3.1 illustrates the graph of $F_N(T)$ against T for $N = 0,1,2, \ldots, 6$ when p and $\hat{p}$ are zero and h is strictly positive.

3.8 Treatment of Back Order Regime

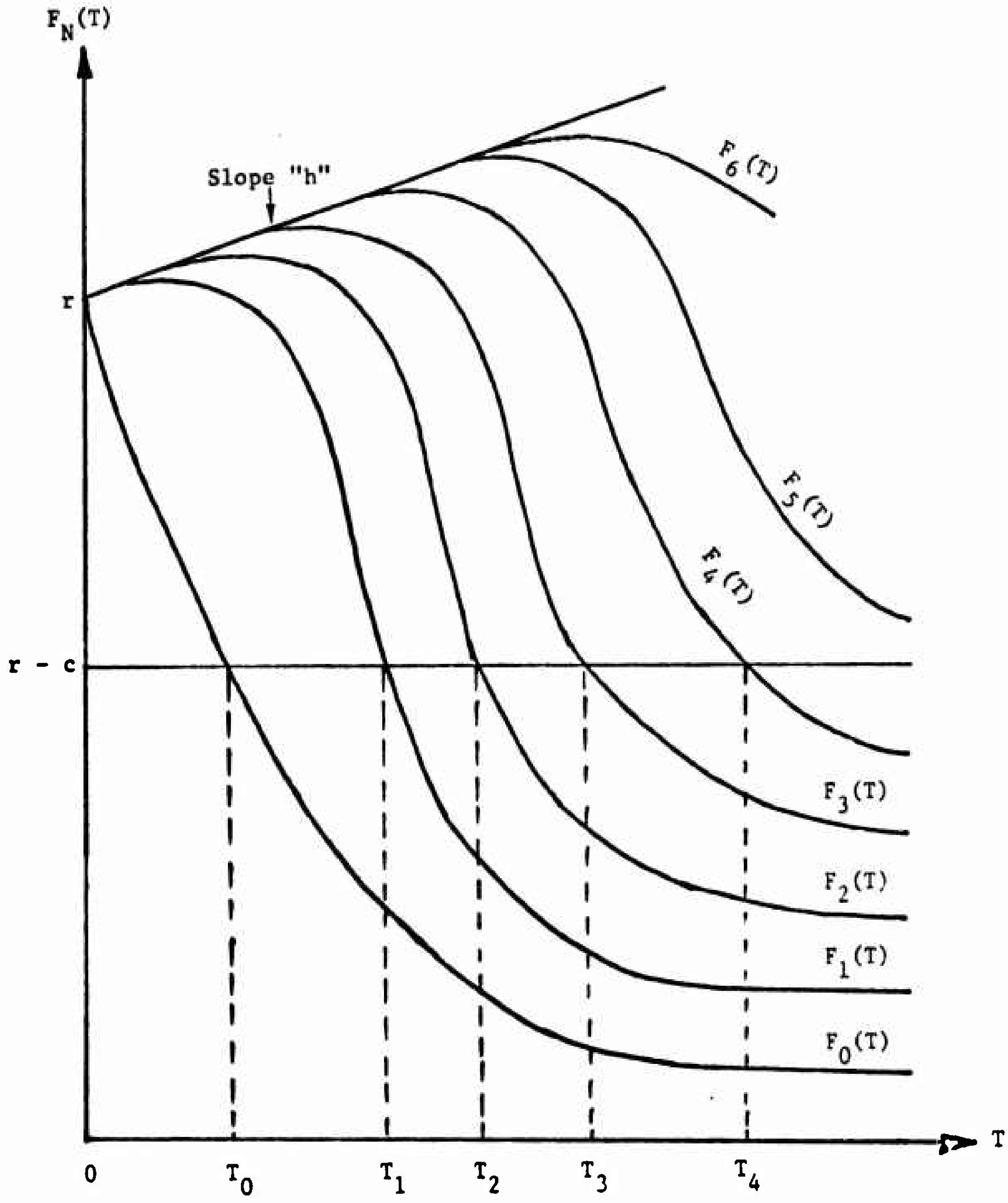
It was pointed out in Section 3.0 that one can follow the policy of filling up the back orders at the end of period. Since it is a single-period problem, the revenue from filling the back orders should be included in the revenue expression. This means an additional revenue equals $(r - c)$ times the amount back ordered, neglecting the set-up cost k for reordering. Thus, the net return function will be

$$\pi(y,T) = rm(T) - k\delta(y - 0) - G(y,T)$$
$$+ (r - c) \sum_{n=y+1}^{\infty} (n - y)P_n(T) .$$

The above expression may be directly obtained from (2.4) by simply replacing the parameter "p" by "(p + c - r)." Thus, all the results of optimal order policy and its properties, given in Chapter 2, follow identically with the replacement of "p" by "(p + c - r)."

Optimal Review Period

Though the analysis of Chapter 2 went without any hitch, it does not



$F_N(T)$ against T (Lost Sales Case with $p = \hat{p} = 0$)

FIGURE 3.1

happen so, for finding an optimal review period. The place where it differs is in the limit of the order level optimized revenue function as $T \to \infty$.
In other words,

$$\lim_{T\to\infty} \pi(T) = -\infty \text{ is not true always.}$$

Referring to Section 3.3, and replacing "p" by "p + c - r", the expression for $\pi(T)$ may be written as,

$$\pi(T) = \frac{1}{\bar{p}} (Q_1 + Q_2 + Q_3 + Q_4)$$

where

$$Q_1 = [r - (p + c)]\rho\bar{q} - k\bar{p}$$

$$Q_2 = \bar{p}\left[py_o(T) - (p + c) \sum_{n=0}^{y_o(T)} [y_o(T) - n]P_n(T)]\right]$$

$$Q_3 = -\bar{p}\left[\frac{h}{\alpha} \sum_{n=0}^{y_o(T)} [y_o(T) - n] \frac{I_{\bar{q}}(n + 1,\rho)}{\rho + n}\right]$$

$$Q_4 = -\left[\hat{p}/\alpha \sum_{n=y_o(T)+1}^{\infty} \frac{[n - y_o(T)]I_{\bar{q}}(n + 1,\rho)/\rho + n}{1/\bar{p}}\right].$$

As $T \to \infty$, the numerator $[Q_1 + Q_2 + Q_3 + Q_4]$ tends to $\rho[r - (p + c)] - \rho\hat{p}/\alpha$.
(Refer to Section 3.3.) Hence,

$$\lim_{T\to\infty} \pi(T) = \begin{cases} +\infty & \text{if } \lambda[r - (p + c)] - \rho\hat{p} \geq 0 \\ -\infty & \text{if } \lambda[r - (p + c)] - \rho\hat{p} < 0 . \end{cases}$$

Thus, there are two cases to examine depending on the values of the parameters. The expression for $\pi_N'(T)$ may also be written from (3.10) and

(3.11), with the substitution $p = (p + c - r)$. Thus,

$$\pi_N'(T) = \frac{\lambda[r - (p + c)] - \rho\hat{p} + S_N(T)}{\bar{p}} \tag{3.24}$$

where

$$S_N(T) = \bar{p}\Bigg[(N + \rho)\hat{p} - (h + \hat{p}) \sum_{n=0}^{N} (N - n)P_n(T)$$

$$- (p + c) \sum_{n=0}^{N} [N - n][(\rho + n - 1)\bar{p}\alpha P_{n-1}(T) - \lambda P_n(T)]\Bigg] \tag{3.25}$$

$$\text{for all } T_{N-1} \leq T < T_N .$$

Comparing (3.11) and (3.24), it is clear that Propositions 3.3 thru 3.7 hold for this case too. Also, (3.18) holds, i.e.,

$$S_N(T) > 0 \qquad \text{for all } T_{N-1} \leq T < T_N \qquad \text{and } N = 1,2, \ldots, \bar{N} . \tag{3.26}$$

If $\lambda[r - (p + c)] - \rho\hat{p} < 0$, then the results of Section 3.5 hold and the algorithm to compute the optimal review period, as given in Section 3.6, may be applied without modification. On the other hand, if $\lambda[r - (p + c)] - \rho\hat{p} \geq 0$, then

$$\pi_N'(T) > 0 \qquad \text{for all } T \in [T_0,\infty) .$$

(See (3.24) and (3.26).) Hence, $\pi(T)$ increases indefinitely as T increases and the optimal review period may be made as large as possible. This case may not happen in a number of systems as "c" may be as high as "r," if the product is bought from a competitor so as to satisfy the customers, and the back order cost "p" in general is positive.

Since seasonal goods have a finite upper bound for the length of the review period (i.e., the length of the season), even if $\lambda[r - (p + c)] - \rho\hat{p} \geq 0$, the optimal review period can only be made as large as the length of the season.

CHAPTER 4

ESTIMATION OF PARAMETERS

4.1 The Estimation Problem

As discussed in Chapter 1, the probability distribution of demand is given by

$$P_n(T) = \frac{\Gamma(\rho + n)}{\Gamma(\rho)\Gamma(n + 1)} (e^{-\alpha T})^{\rho}(1 - e^{-\alpha T})^n$$

where $P_n(T)$ = probability of n demands in $[0,T]$ and $\rho = \lambda/\alpha > 0$. Once the parameters λ (the constant demand rate) and α (the unit contagion rate) are determined, $P_n(T)$ is known for given T, and the results of Chapter 2 and 3 may be applied to find optimal order policy and review period. Thus, we are left with estimating the values of α and λ using the knowledge of the demands in the previous periods. Of course, this poses a problem for the first period as there is no prior knowledge. This can be overcome if a sample survey is done where the product was introduced in a smaller scale in a sample area. As a matter of fact, the practice of conducting a pilot study is prevelant in many cases. For example, a cereal manufacturer usually introduces the product in a small sample area before it is introduced nationwide. Using the same estimation procedures described in later sections, a good estimate of the initial values of α and λ may be obtained.

A number of estimation procedures are available in literature like the method of moments, method of frequencies, maximum likelihood procedure, modified chi-square method and so on. We shall essentially describe two methods and exhibit their estimators. The first method will be the direct and simple method of moments which may not be efficient. But the second method will be the maximum likelihood estimate which we will show for large sample size, that it is asymptotically unbiassed, minimum variance and efficient estimator of the parameters.

4.2 Method of Moments

This is the oldest general method proposed for estimating the values of the parameters of a distribution by means of a set of sample values. This method consists in equating a convenient number of the sample moments to the corresponding moments of the distribution, which are functions of the unknown parameters. By considering as many moments as there are parameters to be estimated and solving the resulting equation with respect to the parameters, estimates of the latter are obtained. This method often leads to comparatively simple calculations in practice.

Assume we have N sample areas and random variables $X_1, X_2, \ldots, X_N$ denote the number of demands in $[0,T]$ from the ith sample. X_i's are independent, identically distributed and have a contagious distribution such that

$$P_{X_i=x}(T) = \frac{\Gamma(\rho + x)}{\Gamma(\rho)\Gamma(x + 1)} (e^{-\alpha T})^{\rho} (1 - e^{-\alpha T})^{x} \quad \text{for all } x = 0,1,2. \ldots \tag{4.1}$$

Define:

$$\text{Sample Mean} = \bar{X} = \frac{\sum_{i=1}^{N} X_i}{N}$$

$$\text{Sample Variance} = s^2 = \frac{\sum_{i=1}^{N} (X_i - \bar{X})^2}{N} .$$

We know (from Chapter 1) that the theoretical moments are given by,

$$m(T) = \text{expectation} = \frac{\rho(1 - e^{-\alpha T})}{e^{-\alpha T}}$$

$$V(T) = \text{variance} \quad = \frac{\rho(1 - e^{-\alpha T})}{e^{-2\alpha T}} .$$

Equating the theoretical and sample moments,

$$(4.2) \qquad m(T) = \frac{\sum_{i=1}^{N} X_i}{N} = \bar{X} .$$

$$(4.3) \qquad V(T) = \frac{\sum_{i=1}^{N} (X_i - \bar{X})^2}{N} = s^2 .$$

Dividing (4.2) by (4.3) we get, $\hat{\alpha}$ the estimator of α as,

$$e^{-\hat{\alpha}T} = \frac{\bar{X}}{s^2} , \text{ i.e.,}$$

$$\hat{\alpha} = \frac{1}{T} \log_e \left(\frac{\sum_{i=1}^{N} (X_i - \bar{X})^2}{\sum_{i=1}^{N} X_i} \right).$$

Substituting $\hat{\alpha}$ in (4.2), we get the estimator for λ as

$$(4.5) \qquad \hat{\lambda} = \frac{\hat{\alpha}\bar{X}^2}{s^2 - \bar{X}} .$$

Thus, $\hat{\alpha}$ and $\hat{\lambda}$ are the estimators for α and λ and the actual estimates can be found by replacing the random variables $X_1, X_2, \ldots, X_N$ by observed values $x_1, x_2, \ldots, x_N$ in expressions (4.4) and (4.5) respectively.

4.3 The Method of Maximum Likelihood[†]

From a theoretical point of view this is the most important general method of estimation.

Using the same notations of Section 4.2, we define the likelihood function L of the sample of n values from a population of the discrete type by the relation

$$L(X_1, \ldots, X_n; \alpha, \lambda) = P_{x_1}(\alpha,\lambda) \ldots P_{x_n}(\alpha,\lambda) . \tag{4.6}$$

When the sample values of $X_1, \ldots, X_n$ are given, the likelihood function L becomes a function of the two variables α and λ. The method of maximum likelihood now consists in choosing, as an estimate of the unknown pair of values (α,λ), the particular value (α,λ) that maximizes the likelihood function. Since $\log L$ attains its maximum for the same value of (α,λ) as L, we thus have to solve the likelihood equation

$$\frac{\delta \log L}{\delta \alpha} = 0 \tag{4.7}$$

$$\frac{\delta \log L}{\delta \lambda} = 0 \tag{4.8}$$

with respect to α and λ. We shall disregard any root of the form α = constant and/or λ = constant, thus counting as a solution only a root which effectively depends on the sample values $X_1, \ldots, X_n$. Any

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[†]For a detailed discussion the reader is referred to Cramér [2].

solution of the likelihood equation will then be called a maximum likelihood estimate of (α,λ) .

The importance of the method is clear by the following two propositions:

(i) If a pair of joint efficient estimates α^* and λ^* exists, the likelihood equations will have the unique solution $\alpha = \alpha^*$ and $\lambda = \lambda^*$.

(ii) Any solution of (4.7) and (4.8) is a sufficient estimate of (α,λ) if a sufficient estimate exists. From (4.6),

$$\log_e L = \sum_{i=1}^{n} \log_e P_{x_i}(T) ,$$

and from (4.1)

$$\log P_{x_i}(T) = \log \Gamma(\rho + x_i) - \lambda T + x_i \log (1 - e^{-\alpha T}) - \log \Gamma(\rho) - \log \Gamma(x_i + 1) .$$

Thus,

$$\log_e L = \sum_{i=1}^{n} \log \Gamma(\rho + x_i) - n\lambda T + n\bar{x} \log (1 - e^{-\alpha T}) - n \log \Gamma(\rho) - \sum_{i=1}^{n} \log \Gamma(x_i + 1) .$$

Define

$$\psi(y) \triangleq \frac{d \log \Gamma(y)}{dy} \triangleq \frac{\Gamma'(y)}{\Gamma(y)} \tag{4.10}$$

$\psi(y)$ is known in literature as digamma function and tables are available in Fletcher, et al (refer Appendix III on Gamma functions). Setting $\frac{\delta \log L}{\delta \lambda} = 0$ and using (4.10), we get

$$\frac{1}{\alpha}\sum_{i=1}^{n} \psi(\rho + x_i) - nT - \frac{n}{\alpha}\psi(\rho) = 0 . \tag{4.11}$$

Similarly,

$$\frac{\delta \log L}{\delta \alpha} = -\frac{\lambda}{\alpha^2}\sum_{i=1}^{n} \psi(\rho + x_i) + \frac{n\bar{X}Te^{-\alpha T}}{1 - e^{-\alpha T}} + \frac{n\lambda}{\alpha^2}\psi(\rho) = 0 .$$

Using (4.11), this reduces to

$$\frac{\lambda}{\alpha} = \frac{\bar{x}e^{-\alpha T}}{1 - e^{-\alpha T}} ,$$

i.e.,

$$\lambda(e^{\alpha T} - 1) = \alpha\bar{x} . \tag{4.12}$$

Now we are left with two simultaneous equations in two unknowns. Of course, a direct approach is to solve (4.11) and (4.12) iteratively for a pair (α,λ) . This should not present any problem once it is committed to a digital computer.

4.4 Efficiency of the Maximum Likelihood Estimator

We shall show a few properties about the contagious distribution given in (4.1) which are necessary to show that the pair of values (α,λ) obtained from (4.11) and (4.12) is asymptotically unbiassed, efficient estimate of (α,λ) as the number of samples is large.

P1:

The first, second and third derivatives of $\log P_n(T)$ with respect to α and λ exist.

Proof:

The truth of this statement follows from the fact that Gamma function has continuous derivatives of all orders (refer Appendix III).

P2:

The contagious distribution $\{P_n(T)\}$ has finite moments of all orders, for given T .

Proof:

For given T , the generating function of $P_n(T)$ is given by,

$$G(Z) = \left(\frac{\bar{p}}{1 - \bar{q}Z}\right)^{\rho} \quad \text{where} \quad \bar{p} = e^{-\alpha T} \, ,$$

$$\bar{p} + \bar{q} = 1 \, .$$

Differenting this with respect to Z ,

$$G'(Z) = \frac{\bar{p}^{\rho}\bar{q}\rho}{(1 - \bar{q}Z)^{\rho+1}} \, .$$

$$\text{The first moment} = \sum_{n=0}^{\infty} n \, P_n(T) = \operatorname*{Lt}_{Z\to 1} G'(Z)$$

$$= \frac{\rho\bar{q}}{\bar{p}} \, .$$

Similarly,

$$G''(Z) = \frac{\rho(\rho + 1)\bar{q}^2\bar{p}^\rho}{(1 - \bar{q}Z)^{\rho+2}}$$

and $G''(1) = \sum_{n=0}^{\infty} n(n - 1)P_n(T) = \frac{\rho(\rho + 1)\bar{q}^2}{\bar{p}^2}$. Thus, in general,

$$G^n(1) = \underset{Z\to 1}{\mathrm{Lt}} \frac{d^n G(Z)}{dZ^n} = \sum_{n=0}^{\infty} n(n - 1) \ldots 2.1 \cdot P_n(T)$$

$$= \frac{\rho(\rho + 1) \ldots (\rho + n - 1)\bar{q}^n}{\bar{p}^n} .$$

Thus, $P_n(T)$ has finite moments of all orders for $T \epsilon [0,\infty)$.

<u>P3</u>:

$$\sum_{n=0}^{\infty} \Gamma(\rho + n)P_n(T) \text{ is finite and positive.}$$

<u>Proof</u>:

It is clear that the summation is positive as $\Gamma(\rho + n) > 0$ for all $n = 0,1,2, \ldots$ and $\rho > 0$. Finiteness follows from <u>P2</u>.

The reader can verify himself that Cramér's conditions for asymptotic efficiency of maximum likelihood estimators follow directly from Properties P1, P2 and P3. Thus, the maximum likelihood estimator $(\hat{\lambda},\hat{\alpha})$ given by (4.11) and (4.12) is asymptotically unbiassed efficient estimator.

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APPENDIX I

SOLUTION OF ORDINARY DIFFERENTIAL EQUATION

Reference:

Granville, W.A., P.F. Smith and W.R. Longley, ELEMENTS OF THE DIFFERENTIAL AND INTEGRAL CALCULUS, Ginn and Company, San Francisco, p. 380, (1934).

Here we give a method to solve ordinary differential equations of the form

$$\frac{dy}{dx} + Py = Q \tag{1}$$

where P and Q are functions of x alone or constants.

To integrate (1), let

$$y = uz \tag{2}$$

where z and u are functions of x to be determined. Differentiating (2),

$$\frac{dy}{dx} = u\frac{dz}{dx} + z\cdot\frac{du}{dx} . \tag{3}$$

Substituting from (3) and (2) in (1), we get

$$u\frac{dz}{dx} + z\frac{du}{dx} + Puz = Q ,$$

i.e.,

$$u\frac{dz}{dx} + \left(\frac{du}{dx} + Pu\right)z = Q . \tag{4}$$

We now determine u by integrating

$$\frac{du}{dx} + Pu = 0 ,$$

in which the variables x and u are separable. Using the value of u thus obtained, we find z by solving

$$u \frac{dz}{dx} = Q ,$$

in which x and z can be separated. Obviously, the values of u and z thus found will satisfy (4), and the solution of (1) is then given by (2).

The differential Equation (1.5) we had in Chapter 1, is given by

$$(5) \qquad P_n'(t) = -(\lambda + \alpha n)P_n(t) + [\lambda + \alpha(n - 1)]P_{n-1}(t) .$$

By putting $y = P_n(t)$, $x = t$, $P = (\lambda + \alpha n)$,

$$Q = \lambda + \alpha(n - 1) ,$$

we will have the same form as in (1). Hence, we use the method described just now to solve for $P_n(t)$.

APPENDIX II

PROPERTIES OF BETA FUNCTION

By definition, a Beta function of two parameters m , n is given by

$$B(m,n) = \int_0^1 x^{m-1}(1 - x)^{n-1}dx \quad \text{where} \quad \begin{array}{l} m > 0 \\ n > 0 \end{array}$$

$$= 2 \int_0^{\pi/2} \text{Sin}^{2m-1}(\theta)\ \text{Cos}^{2n-1}(\theta)d\theta \ .$$

Also,

$$B(m,n) = B(n,m) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m + n)} \ .^{\dagger}$$

An incomplete Beta function is defined by,

$$B_x(m,n) = \int_0^x x^{m-1}(1 - x)^{n-1}dx \quad \text{where} \quad \begin{array}{l} m > 0 \\ n > 0 \ . \end{array}$$

The ratio of incomplete Beta function to the (complete) Beta function is known as the incomplete Beta function ratio and is denoted by

(1) $$I_x(m,n) = \frac{B_x(m,n)}{B(m,n)} \ .$$

Note that $I_1(m,n) \equiv 1$. Also,

(2) $$I_x(m,n) = 1 - I_y(m,n) \quad \text{where} \quad x + y = 1 \ .$$

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† A proof of this relation can be found in Cramér [2].

Tables of incomplete Beta functions and ratios are available in Pearson [11].

It can be verified† using (1) and (2) when m is an integer, that

$$I_x(m,n) = 1 - (1 - x)^n \sum_{i=0}^{m-1} \binom{n-1+i}{i} x^{i-1} .$$

The advantage of this equation is that of computational feasibility. The above series is made up of entirely positive terms and hence can be summed quite accurately, even for fairly large values of parameters m and n .

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†The actual derivation is given in Harter, H.L., NEW TABLES OF INCOMPLETE GAMMA FUNCTION, ETC., Aerospace Research Laboratories, U.S. Government Printing Office, Washington, D.C. 20402, (1964).

APPENDIX III

THE GAMMA FUNCTION

Reference:

Cramér, H. [2].

The Gamma function $\Gamma(p)$ is defined for all real $p > 0$ by the integral

(1) $$\Gamma(p) = \int_0^\infty x^{p-1}e^{-x}dx \ .$$

The function is continuous and has continuous derivatives of all orders:

(2) $$\Gamma^{(r)}(p) = \int_0^\infty x^{p-1}(\log x)^r e^{-x}dx$$

for any $p > 0$. When p tends to 0 or to $+\infty$, $\Gamma(p)$ tends to $+\infty$. Since the second derivative is always positive, $\Gamma(p)$ has one single minimum in $(0,\infty)$. Approximate calculation shows that the minimum is situated in the point $p_o = 1.4616$, where the function assumes the value $\Gamma(p_o) = 0.8856$.

By a partial integration, we obtain from (1) for any $p > 0$

$$\Gamma(p + 1) = p\Gamma(p) \ .$$

When p is equal to a positive integer n , a repeated use of the last equality gives, since $\Gamma(1) = 1$,

$$\Gamma(n + 1) = n \ !$$

From (1), we further obtain the relation

$$\int_0^\infty x^{\lambda-1} e^{-\alpha x} dx = \frac{\Gamma(\lambda)}{\alpha^\lambda}$$

where $\alpha > 0$, $\lambda > 0$. From (2), when $r = 1$,

$$\Gamma'(p) = \int_0^\infty x^{p-1} (\log x) e^{-x} dx \ .$$

Define $\psi(p) = \frac{d\log\Gamma(p)}{dp} = \frac{\Gamma'(p)}{\Gamma(p)}$. $\psi(p)$ is called digamma function. Tables of digamma functions are available in Fletcher, A. et al, AN INDEX OF MATHEMATICAL TABLES I, Sect. 14.4, Addison-Wesley Publishing Co., Inc., Reading, Massachusetts, (1962).

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